Two general frameworks for sequential change detection with composite, nonparametric classes



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# **Outline of this lecture series**

Yesterday: game-theoretic testing



3. Now: game-theoretic change detection

### Super high-level problem setup

You observe data (scalars or vectors or objects) one at a time.

You calculate **some statistic** of the data as you go along. ("CUSUM", "SR")

When the statistic crosses **a threshold**, you proclaim a change.

<u>Aim</u>: Minimize <u>detection delay</u> while controlling <u>false alarms</u>.

### **Metrics of success**

#### Average run length (ARL) or Frequency of False Alarms: this is the analog of "Type-I error" from testing

<u>Definition</u>: When there is no change point, the ARL is the expected number of steps before we (falsely) proclaim a change point.

Eg: maybe we are ok with a false alarm roughly every 1000 steps.

#### Detection delay (DD): analog of "Type-II error"

<u>Definition</u>: When there is a change point at some time  $\nu$ , DD is the expected number of steps after  $\nu$  that we need to (correctly) proclaim a change point.

### Critical issue in practice

#### Controlling the ARL is very hard!

If the pre-change distribution is perfectly known, then controlling ARL is easy: set threshold via offline **simulation**, or use **analytical calculations** (asymptotics) to find threshold.

Cannot do this if the pre-change data is only known to lie in some set of distributions

i.e. we only know some partial aspects of pre-change distribution, (Eg: we know variance < 10, but not sure what the exact value is)

If we do not know how to model some aspects of the pre-change distribution (i.e. "nonparametric") (Eg: we know the data is bounded, but we don't know much else) In such cases, it is much harder to control the ARL. Simulation and math both don't easily work.

### Two recent reductionist works

# Using "E-detectors"

#### (NEJSDS'23)

When pre- and post-change distributions are known to lie in separate classes  $\mathscr{P}, \mathscr{Q}$ (eg: mean change from < 0 to > 0)

<u>Key idea</u>: reduce change detection to repeated sequential testing of  $\mathscr{P}$  vs.  $\mathscr{Q}$ 

<u>Method</u>: at every time t, start a new level- $\alpha$  sequential test based on  $X_t, X_{t+1}, \ldots$  and declare change when accumulated evidence crosses  $1/\alpha$ 

#### <u>Main theorem</u>: ARL $\geq 1/\alpha$

## Via "confidence sequences"

#### (arXiv)

When pre- and post-change distributions are unknown but from the same class (eg: the mean changed from a to b)

Key idea: reduce change detection to repeated sequential estimation

<u>Method</u>: at every time t, start a new level- $\alpha$  confidence sequence based on  $X_t, X_{t+1}, \ldots$  and declare change when their intersection is empty.

<u>Main theorem</u>: ARL  $\geq 1/\alpha$ 

Main theorem2: "Optimal" detection delays Main theorem2: "Optimal" detection delays

<u>Advantage</u>: sequential testing/estimation are basic problems, increasingly well studied.



- I. Reducing <u>partitioned</u> change detection to <u>sequential testing</u>
- 2. Reducing <u>non-partitioned</u> change detection to <u>sequential estimation</u>
- 3. High level summary of game-theoretic statistics

### **E-detectors**



Jaehyeok Shin (Google)



Alessandro Rinaldo (CMU / UT Austin)

TLDR: Reducing sequential (nonparametric) change detection to (modern) sequential testing

### Recap: Lorden's reduction to testing (1970)

What is a "one-sided" level- $\alpha$  sequential test of  $\mathscr{P}$  vs.  $\mathscr{Q}$ ?  $H_0: X_1, X_2, \ldots \sim P \in \mathscr{P}$  vs.  $H_1: X_1, X_2, \ldots \sim Q \in \mathscr{Q}$ 

For each  $t \ge 1$ , after seeing  $X_t$  we output 0 or 1. 0 means "not enough evidence to reject  $\mathscr{P}$ , collect more data". I means "we have enough evidence to reject  $\mathscr{P}$ ".

A level- $\alpha$  sequential test  $\phi$  satisfies  $P(\exists t \in \mathbb{N} : \phi(X_1, ..., X_t) = 1) \leq \alpha$  for all  $P \in \mathscr{P}$ .

Lorden's Reduction: at every time t, start a *new* level  $\alpha$  sequential test based on  $X_t, X_{t+1}, \ldots$  (so there are t tests running after t steps). We declare a change when any of them rejects the null.

<u>Key theorem</u>: This method controls ARL (FFA) at level  $1/\alpha$ .

TLDR of our paper: more sophisticated reduction to sequential testing. Idea: accumulate the evidence across tests.

### Quantifying evidence using e-processes

Consider sequential testing  $H_0: P \in \mathscr{P}$ 

An e-process for  $\mathscr{P}$  is a nonnegative process  $\Lambda \equiv (\Lambda_n)_{n \geq 1}$ such that  $\mathbb{E}_P[\Lambda_{\tau}] \leq 1$  for all  $P \in \mathscr{P}, \tau \in \mathscr{T}$ .

We reject the null whenever the "evidence"  $\Lambda_t$  exceeds  $1/\alpha$ .

E-processes are nonparametric, composite generalizations of likelihood ratios and have a nice game-theoretic foundation.

Vovk, Shafer, Grunwald, R, Wang, Larsson, Koolen, Ruf, Howard, etc. (last 5 years)

Define the SR e-detector as  $M_n^{SR} := \sum_{j=1}^n \Lambda_n^{(j)}$ , where  $\Lambda^{(j)}$  is a  $\mathscr{P}$ -e-process started at time j.

#### **E-detector**

 $\mathscr{P}$ : pre-change set of distributions over infinite sequence  $\mathscr{F} \equiv (\mathscr{F}_n)_{n \geq 1}$ : filtration, and  $\mathscr{F}_n$  need not be  $\sigma(X_1, \dots, X_n)$ .

An e-detector for  $\mathscr{P}$  is a nonnegative process  $M \equiv (M_n)_{n \geq 1}$ such that  $\mathbb{E}_P[M_{\tau}] \leq \mathbb{E}_P[\tau]$  for all  $P \in \mathscr{P}, \tau \in \mathscr{T}$ .

 $\mathcal{T}$ : set of all stopping times with respect to  $\mathcal{F}$ 

### Nonasymptotic ARL control

Define stopping time  $N_{1/\alpha}^* := \inf\{t \ge 1 : M_t \ge 1/\alpha\}$ .

<u>Theorem</u>: If *M* is an e-detector for  $\mathscr{P}$ , then the ARL is  $\mathbb{E}_P[N^*_{1/\alpha}] \ge 1/\alpha \ \forall P \in \mathscr{P}$ 

# Computationally efficient e-detectors using "baseline" e-processes

Often, our e-process can be written as 
$$\Lambda_n^{(j)} := \prod_{i=j}^n L_i$$
  
where  $\mathbb{E}_P[L_n | \mathscr{F}_{n-1}] \leq 1$  for all  $P \in \mathscr{P}$ .

Then,  $M_n^{SR} := L_n(M_{n-1}^{SR} + 1)$  can be computed online, like the Shiryaev-Roberts procedure.

Example: likelihood ratios, if  $\mathscr{P} = \{P\}$  is a point null, but also any setting with a nonnegative supermartingale for composite  $\mathscr{P}$ .

# Cleveland Cavaliers (NBA, 2011-18)

#### Plus-minus = Points scored - Points conceded

**Plus-Minus of the Cavaliers** 



Game Date

Is there a changepoint from a negative to a positive plus-minus?

E-detector: a nonparametric framework to answer such questions.

The paper: related work (Lorden, Pollak, Siegmund, Tartakovsky, Veeravalli, Xie, Harchaoui, and many others)

# Why is it hard?

Challenges: the plus-minus stat of a game is a bounded r.v. between [-100, +100] (let's rescale to [0,1]) and its mean varies over time (with form, injuries, etc), and we know nothing else about the distribution.

 $X_1, X_2, \ldots \sim P$  for some  $P \in \mathscr{P}$ 

where  $\mathscr{P} := \{ P \text{ on } [0,1]^{\infty} : \mathbb{E}[X_n | \text{past}] \le 1/2 \text{ for all } n \ge 1 \}$ 

If there is a change at time  $\nu$ , then  $X_{\nu}, X_{\nu+1}, \dots \sim Q$  for some  $Q \in Q$ where  $Q := \{Q \text{ on } [0,1]^{\infty} : \mathbb{E}[X_n | \text{past}] > 1/2 \text{ for all } n \geq 1\}$ 

Here P, Q are distributions on *infinite* sequences of observations, so the data are not iid (neither i nor id), and are only restricted by their conditional means

### Method

# $$\begin{split} X_1, X_2, \dots &\sim P \text{ for some } P \in \mathscr{P} \\ \text{where } \mathscr{P} := \{P \text{ on } [0,1]^\infty : \mathbb{E}[X_n | \text{ past}] \leq 1/2 \text{ for all } n \geq 1 \} \end{split}$$

If there is a change at time  $\nu$ , then  $X_{\nu}, X_{\nu+1}, \dots \sim Q$  for some  $Q \in Q$ where  $Q := \{Q \text{ on } [0,1]^{\infty} : \mathbb{E}[X_n | \text{past}] > 1/2 \text{ for all } n \geq 1 \}$ 

Fix Q for now: we have seen that we can mix over Q, or plug-in.

# For a fixed $Q_t$ , recall the previous two lectures: $\prod_{i=1}^{t} (1 + \lambda_Q^*(X_t - 1/2))$

Is the optimal supermartingale. So plug this into the e-detector.

### Result





Our e-detector announces a change point during the 2014-15 season. It controls the average run length (ARL) at  $1/\alpha = 1000$ , which is more than twelve seasons of games. More broadly: the e-detector can help detect changes when the pre-change and post-change distributions are composite, non-stationary and nonparametrically specified (eg: no common reference measure, no likelihoods, infinite dimensional nuisance parameters, etc.)

(Simulation, asymptotics are intractable for ARL.)

# A short list of e-processes for other settings

• ANY  $\mathscr{P}$  for which max-likelihood (after smoothing or profiling) is feasible (eg: shape constraints like monotone, log-concave)

Wasserman, Balakrishnan, Ramdas'20 Universal inference

•  $\mathcal{P} = \{P : X_1, X_2, \dots \text{ are exchangeable}\}, Q \text{ unrestricted}$ 

Vovk'21 (Testing randomness online) Ramdas, Ruf, Larsson, Koolen'21 (Testing exchangeability: fork-convex hulls, supermartingales and e-processes)

•  $\mathscr{P} := \{(P \times P)^{\infty} : \text{ for any } P\}, \ \mathscr{Q} := \{(P \times Q)^{\infty} : \text{ for any } P \neq Q\}$ 

Shekhar, Ramdas'21 (Nonparametric two-sample testing by betting)

### Some more e-processes for other settings

$$\mathcal{P} := \{ P^{\infty} : \mathbb{E}[X] \le \mu, \forall \operatorname{ar}(P) \le \sigma^2 \}$$
$$\mathcal{Q} := \{ P^{\infty} : \mathbb{E}[X] > \mu, \forall \operatorname{ar}(P) \le \sigma^2 \}$$

Wang, Ramdas'23 (Catoni-style CSs for heavy-tailed mean estimation)

•  $\mathscr{P} := \{P^{\infty} : \mathbb{E}[X] \le \mu, \forall ar(P) \le \sigma^2, adversary corrupts \epsilon \text{ fraction of data} \}$  $\mathscr{Q} := \{P^{\infty} : \mathbb{E}[X] > \mu, \forall ar(P) \le \sigma^2, adversary corrupts \epsilon \text{ fraction of data} \}$ 

Wang, Ramdas'23 (Huber-robust confidence sequences)

• 
$$\mathscr{P} := \{P_{XY}^{\infty} : \text{where } P_{XY} = P_X \times P_Y\},\ \mathscr{Q} := \{P_{XY}^{\infty} : \text{where } P_{XY} \neq P_X \times P_Y\}$$

Shekhar, Ramdas'23 and Podkopaev et al.' 23 (Nonparametric independence testing by betting)

### Detection delay: suppose Q is known

For a changepoint stopping rule N, Define worst detection delay  $\mathscr{D}(N) := \sup_{\nu \geq 0} \mathbb{E}_{P,\nu,Q}[(N - \nu)_+ | N > \nu]$ 

If the baseline increment  $L_n$  is function of only  $X_n$ and the post-change observations from Q are strongly stationary, then  $\mathscr{D}(N^{SR}) \leq \frac{\log(1/\alpha)}{\mathbb{E}_Q \log L_1} + \frac{\mathbb{E}_Q \log^2 L_1}{(\mathbb{E}_Q \log L_1)^2} + 1$ ,

where  $N^{SR} := \inf\{t \ge 1 : M_t^{SR} \ge 1/\alpha\}.$ 

Minimizing delay corresponds to maximizing  $K := \mathbb{E}_Q \log L_1$ , but we usually don't know Q.

Let  $(L^{\lambda})_{\lambda \in \Pi}$  be a family of baseline processes indexed by  $\lambda$ .  $\lambda^* := \underset{\lambda \in \Pi}{\operatorname{arg\,max}} \mathbb{E}_Q \log L_1^{\lambda}$  depends on Q.

### **Mixtures of e-detectors**

A convex combination (mixture) of e-detectors (ARL control is also an e-detector. maintained)

This fact can be used to take statistically and computationally efficient mixtures over  $\Pi$  (implicitly over Q), in order to derive e-detectors that adapt to unknown Q.

We derive e-detectors where the number of mixture components increases (logarithmically) with time.

### **Detection delay for the mixture e-SR procedure**

$$\mathcal{D}(N^{mSR}) \leq \frac{g_{\alpha}}{D(Q \| \mathcal{P})} + \frac{\mathbb{V}_Q[\log L_1^{\lambda^*}]}{D^2(Q \| \mathcal{P})} + 1$$

Defining  $\Delta^* = \nabla \psi(\lambda^*)$  as the "signal strength", lying in  $[\Delta_L, \Delta_U]$ , we have  $g_{\alpha} \leq \inf_{\eta > 1} \eta \left[ \log(1/\alpha) + \log\left(1 + \lceil \log_{\eta} \frac{\psi^*(\Delta_U)}{\psi^*(\Delta_L)} \rceil \right) \right]$ 

Qualitatively similar results in the non-separated case.

### Takeaway messages

A  $\mathscr{P}$ -e-detector is a nonnegative adapted process  $M \equiv (M_n)_{n \geq 1}$ such that  $\mathbb{E}_P[M_{\tau}] \leq \mathbb{E}_P[\tau]$  for all  $P \in \mathscr{P}, \tau \in \mathscr{T}$ .

A  $\mathscr{P}$ -e-process is a nonnegative adapted process  $\Lambda \equiv (\Lambda_n)_{n \geq 1}$ such that  $\mathbb{E}_P[\Lambda_{\tau}] \leq 1$  for all  $P \in \mathscr{P}, \tau \in \mathscr{T}$ .

Eg: the SR e-detector as 
$$M_n^{SR} := \sum_{j=1}^n \Lambda_n^{(j)}$$
,  
where  $\Lambda^{(j)}$  is a  $\mathscr{P}$ -e-process started at time  $j$ .

Thresholding at 1/α controls ARL at 1/α. Mixtures of e-detectors are also e-detectors. Baseline e-processes (eg: exponential) enable online computation.

Now, we can perform changepoint detection in a slew of nonparametric settings, sometimes good bounds on detection delay.



Reducing <u>partitioned</u> change detection to <u>sequential testing</u>

- 2. Reducing <u>non-partitioned</u> change detection to <u>sequential estimation</u>
- 3. High level summary of game-theoretic statistics

### **E-detectors**

(NEJSDS'23)

When pre- and post-change distributions are known to lie in separate classes (eg: mean change from < 0 to > 0)

## Confidence sequences

arXiv'23

When pre- and post-change distributions are simply two different distributions in one class (eg: the mean changed from something to something else)

# Via confidence sequences





Shubhanshu Shekhar (CMU)

TLDR: Reducing sequential (nonparametric) change detection to (modern) sequential estimation

### Sequential Changepoint Detection

Stream of independent  $\mathcal{X}$ -valued observations:  $X_1, X_2, \ldots$ 

- For some  $T \in \mathbb{N} \cup \{\infty\}$ :
  - $X_t \sim P_0$  for  $t \leq T$
  - $\blacktriangleright X_t \sim P_1 \neq P_0 \text{ for } t > T$

Mild requirements on the distributions:

- ▶ Both  $P_0, P_1$  are unknown
- ▶  $P_0, P_1 \in \mathcal{P}$  for some known class of distributions  $\mathcal{P}$
- Decide between

$$H_0: T = \infty$$
, versus  $H_1: T < \infty$ .

- **• Objective:** Define a stopping time  $\tau$  to declare a detection, that
  - $\blacktriangleright$  minimizes false alarms under  $H_0$
  - has a small detection delay,  $( au- au)^+$ , under  $H_1$

A "confidence sequence (CS)" for a parameter  $\theta$ is a sequence of confidence intervals  $(L_n, U_n)$ that are constructed from the first *n* samples, and have a uniform (simultaneous) coverage guarantee.  $\mathbb{P}(\forall n \ge 1 : \theta \in (L_n, U_n)) \ge 1 - \alpha$ .

Equivalently, for any stopping time  $\tau : \mathbb{P}(\theta \in (L_{\tau}, U_{\tau})) \ge 1 - \alpha$ .

Darling, Robbins '67, '70s Lai '76, '84 Robbins, Siegmund '70s

Much stronger than the pointwise (fixed-sample) confidence interval (CI) guarantee:

$$\forall n \geq 1, \mathbb{P}(\theta \in (\tilde{L}_n, \tilde{U}_n)) \geq 1 - \alpha.$$

### Overview of our results

If we can construct a CS for  $\boldsymbol{\theta}$ 

 $\Rightarrow$ 

We can detect changes in  $\theta$ 

Our Reduction: at every time t, start a *new* level- $\alpha$  confidence sequence based on  $X_t, X_{t+1}, \ldots$  (there are t active CSs after t steps). We declare a change when their intersection is empty.

<u>Key theorem I</u>: This method controls ARL (FFA) at level  $1/\alpha$ .

<u>Key theorem 2</u>: The detection delay is minimax optimal for many problems for which the confidence sequence shrinks like  $\widetilde{O}(1/\sqrt{t})$ .

### Main Assumptions

- We work with distribution class  $\mathcal{P} = \{P_{\theta} : \theta \in \Theta\}$ .
- ▶ Possibly infinite dimensional  $\Theta$  endowed with metric *d*.
- $\blacktriangleright P_0 = P_{\theta_0} \text{ and } P_1 = P_{\theta_1} \text{ for } \theta_0, \theta_1 \text{ such that } d(\theta_0, \theta_1) > 0.$

#### Assumptions

1. Uniformly decaying width: We can construct a CS  $\{C_t(\theta) : t \ge 1\}$  for all  $\theta \in \Theta$ , satisfying

 $\sup_{\theta\in\Theta} \sup_{\theta',\theta''\in C_t(\theta)} d(\theta',\theta'') \leq w_t \equiv w_t(\Theta,\alpha),$ 

such that  $\lim_{t\to\infty} w_t = 0$ .

2. Enough pre-change data: Under  $H_1$ , the changepoint T is large enough to ensure  $w_T < \Delta := d(\theta_1, \theta_0)$ .

• Control over the detection delay under  $H_1$ :

► Introduce the "good event":  $\mathcal{E} = \{ \forall t \leq T : \theta_0 \in C_t \}$ 

• If 
$$w_t = \mathcal{O}\left(\sqrt{\log \log t/t}\right)$$
, then  
 $\mathbb{E}[(\tau - T)^+ | \mathcal{E}] = \mathcal{O}\left(\frac{\log \log(1/\Delta)}{\Delta^2}\right)$  where  $\Delta = d(\theta_0, \theta_1)$ .

• Can generalize to arbitrary  $w_t \rightarrow 0$  (next slide).

### Detection Delay Analysis: general w<sub>t</sub>



### Detection Delay Analysis: general w<sub>t</sub>



### Detection Delay Analysis: general $w_t$



### Applications

Mean-shift detection with univariate Gaussians. P = N(0, 1) and P = N(0, 1) with A = 10

 $\blacktriangleright P_{\theta_0} = \mathit{N}(\theta_0, 1), \text{ and } P_{\theta_1} = \mathit{N}(\theta_1, 1), \text{ with } \Delta = |\theta_1 - \theta_0|.$ 

Mean-shift detection with bounded observations
 P<sub>θ0</sub> and P<sub>θ1</sub>, supported on [0, 1] with Δ = |θ1 - θ0|.

#### Changes in CDF

• 
$$\Delta = d_{KS}(\theta_0, \theta_1)$$
, with  $\theta_i = \text{CDF}$  of  $P_{\theta_i}$ .

#### Two-sample changepoint detection

 $\blacktriangleright \ \theta_0 = P \times P, \ \theta_1 = P \times Q, \ \text{and} \ \Delta = \mathsf{MMD}(P, Q).$ 

Several other tasks: distribution shifts in ML, nonparametric regression, exponential family.

# Summary

### Using "E-detectors"

#### arXiv: 2203.03532 (under review)

When pre- and post-change distributions are known to lie in separate classes  $\mathscr{P}, \mathscr{Q}$ (eg: mean change from < 0 to > 0)

Key idea: reduce change detection to repeated sequential testing of  $\mathcal{P}$  vs.  $\mathcal{Q}$ 

<u>Method</u>: at every time t, start a new level- $\alpha$  sequential test based on  $X_t, X_{t+1}, \ldots$  and declare change when accumulated evidence crosses  $1/\alpha$ 

#### <u>Main theorem</u>: ARL $\geq 1/\alpha$

# Using "confidence sequences"

ICML 2023 + to-be-arXived

When pre- and post-change distributions are unknown but from the same class (eg: the mean changed from a to b)

Key idea: reduce change detection to repeated sequential estimation

<u>Method</u>: at every time t, start a new level- $\alpha$  confidence sequence based on  $X_t, X_{t+1}, \ldots$  and declare change when their intersection is empty.

#### <u>Main theorem</u>: ARL $\geq 1/\alpha$

<u>Advantage</u>: sequential testing/estimation are basic problems, increasingly well studied.

# Today's talk

Reducing <u>partitioned</u> change detection to <u>sequential testing</u> Reducing <u>non-partitioned</u> change detection to <u>sequential estimation</u>

3. High level summary of game-theoretic statistics

#### What is game-theoretic statistics?

A subfield whose quest is to quantify uncertainty in statistical inference tasks like

- hypothesis testing
- estimation (confidence sets)
- probabilistic forecasting
- model selection
- change detection

by using game-theoretic intuition, language and formalism.

Helps design **new nonparametric inference procedures** which

- Have strong theoretical guarantees under weaker assumptions,
- Provide greater flexibility in applications,
- Combine the use of prior knowledge with frequentist guarantees,
- Are designed to be <u>adaptive to the underlying data.</u>

# The 4 characters in SAVI



designing "good" ones has been the focus of all my research

for many years.

# E-processes are nonparametric, composite generalizations of likelihood ratios

Every likelihood ratio is a test martingale. Further, every test martingale is (implicitly) a likelihood ratio.

If  $(M_t)$  is a test martingale for  $\mathscr{P}$ , then for every  $P \in \mathscr{P}$ , we can write  $M_t = \frac{Q(X_1, \dots, X_t)}{P(X_1, \dots, X_t)}$  for some Q — it is a "simultaneous likelihood ratio", despite there being no dominating reference measure for  $\mathscr{P}$ .

Test martingales are at the *heart* of parametric inference, (likelihood ratio) — but also nonparametric inference.

E-processes generalize test martingales: they exist more generally.

Ramdas, Ruf, Larsson, Koolen (arXiv:2009.03167) \* : no dominating reference measure for the set of distributions

#### Nontrivial (e-power) test martingales exist

Testing symmetry\* Two-sample testing\* Bounded means\* T-test (in shrunk filtration) Exchangeability\* (in shrunk filtration) Independence testing\* (in shrunk filtration)

#### Nontrivial test supermartingales exist

SubGaussian distributions\* (or any bounded MGF)

Robust, heavy-tailed mean estimation\*

#### Nontrivial test martingales exist

Testing symmetry\* Two-sample testing\* Bounded means\* T-test (in shrunk filtration) Exchangeability\* (in shrunk filtration) Independence testing\* (in shrunk filtration)

#### Nontrivial e-processes exist

Any composite  $\mathscr{P}$  : "universal inference" Exchangeability\* (in original filtration) T-test (in original filtration)

#### Nontrivial test supermartingales exist

SubGaussian distributions\* (or any bounded MGF)

Robust, heavy-tailed mean estimation\*

#### Nontrivial test martingales exist

Testing symmetry\* Two-sample testing\* Bounded means\* T-test (in shrunk filtration) Exchangeability\* (in shrunk filtration) Independence testing\* (in shrunk filtration)

#### Some current and future directions

- I. For a new (nonparametric) problem, how do we design the game and *learn* to bet?)
- 2. When do nontrivial test martingales (not) exist? When do nontrivial test supermartingales (not) exist? When do nontrivial e-processes (not) exist?
- 3. How do we move beyond this tutorial's topics to, say, game-theoretic model selection?
- 4. Everything today was nonasymptotic. What about anytime-valid asymptotics? (Few arXiv preprints)
- 5. How do we tie together game-theoretic statistics with game-theoretic probability?













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Two general frameworks for sequential change detection with composite, nonparametric classes



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