Two general frameworks for sequential change detection with composite, nonparametric classes



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# **Outline of this lecture series**

Yesterday: game-theoretic testing

Today morning: game-theoretic estimation

3. Now: game-theoretic change detection

# Super high-level problem setup

You observe data (scalars or vectors or objects) one at a time.

You calculate **some statistic** of the data as you go along. ("CUSUM", "SR")

When the statistic crosses **a threshold**, you proclaim a change.

Aim: Minimize detection delay while controlling false alarms.

# Metrics of success

#### **Average run length (ARL) or Frequency of False Alarms: this is the analog of "Type-I error" from testing**

Definition: When there is no change point, the ARL is the expected number of steps before we (falsely) proclaim a change point.

Eg: maybe we are ok with a false alarm roughly every 1000 steps.

#### **Detection delay (DD): analog of "Type-II error"**

Definition: When there is a change point at some time *ν*, DD is the expected number of steps after  $\nu$  that we need to (correctly) proclaim a change point.

# Critical issue in practice

#### **Controlling the ARL is very hard!**

If the pre-change distribution is perfectly known, then controlling ARL is easy: set threshold via offline **simulation**, or use **analytical calculations** (asymptotics) to find threshold.

Cannot do this if the pre-change data is only known to lie in some *set* of distributions

i.e. we only know some partial aspects of pre-change distribution, (Eg: we know variance  $< 10$ , but not sure what the exact value is)

If we do not know how to model some aspects of the pre-change distribution (i.e. "nonparametric") (Eg: we know the data is bounded, but we don't know much else) In such cases, it is much harder to control the ARL. Simulation and math both don't easily work.

# Two recent reductionist works

# Using "E-detectors"

#### (NEJSDS'23)

When pre- and post-change distributions are known to lie in separate classes  $\mathscr{P},$ (eg: mean change from  $\langle 0$  to  $> 0$ )

Key idea: reduce change detection to repeated sequential testing of  $\mathscr P$  vs.  $\mathscr Q$ 

Method: at every time t, start a new level- $\alpha$  sequential test based on  $X_{t}, X_{t+1}, \ldots$  and declare change when accumulated evidence crosses 1/*α*

#### Main theorem: ARL  $\geq 1/\alpha$  | Main theorem: ARL  $\geq 1/\alpha$

# Via "confidence sequences"

(arXiv)

When pre- and post-change distributions are unknown but from the same class (eg: the mean changed from *a* to *b*)

Key idea: reduce change detection to repeated sequential estimation

Method: at every time t, start a new level-α confidence sequence based on  $X_{t}, X_{t+1}, \ldots$  and declare change when their intersection is empty.

Main theorem2: "Optimal" detection delays Main theorem2: "Optimal" detection delay

Advantage: sequential testing/estimation are basic problems, increasingly well studied.



- 1. Reducing partitioned change detection to sequential testing
- 2. Reducing non-partitioned change detection to sequential estimation
- 3. High level summary of game-theoretic statistics

# E-detectors



Jaehyeok Shin (Google)



Alessandro Rinaldo (CMU / UT Austin)

TLDR: Reducing sequential (nonparametric) change detection to (modern) sequential testing

# Recap: Lorden's reduction to testing (1970)

What is a "one-sided" level- $\alpha$  sequential test of  $\mathscr{P}$  vs.  $\mathscr{Q}$  ? *H*<sub>0</sub> : *X*<sub>1</sub>, *X*<sub>2</sub>, … ∼ *P* ∈  $\mathcal{P}$  vs. *H*<sub>1</sub> : *X*<sub>1</sub>, *X*<sub>2</sub>, … ∼ *Q* ∈

For each  $t \geq 1$ , after seeing  $X_t$  we output 0 or 1. 0 means "not enough evidence to reject  $\mathcal P$ , collect more data". I means "we have enough evidence to reject  $\mathscr{P}$ ".

A level- $\alpha$  sequential test  $\phi$  satisfies .  $P(\exists t \in \mathbb{N} : \phi(X_1, ..., X_t) = 1) \leq \alpha$  for all  $P \in$ 

Lorden's Reduction: at every time t, start a new level  $\alpha$  sequential test based on  $X_t, X_{t+1}, \ldots$  (so there are t tests running after t steps). We declare a change when any of them rejects the null.

Key theorem: This method controls ARL (FFA) at level 1/*α*.

TLDR of our paper: more sophisticated reduction to sequential testing. Idea: accumulate the evidence across tests.

# **Quantifying evidence using e-processes**

Consider sequential testing  $H_0: P \in \mathcal{P}$ 

An e-process for  $\mathscr P$  is a nonnegative process  $\Lambda \equiv (\Lambda_n)_{n>1}$ such that  $\mathbb{E}_p[\Lambda_{\tau}] \leq 1$  for all  $P \in \mathscr{P}, \tau \in \mathscr{T}$ .

We reject the null whenever the "evidence"  $\Lambda_t$  exceeds  $1/\alpha$ .

E-processes are nonparametric, composite generalizations of likelihood ratios and have a nice game-theoretic foundation.

> Vovk, Shafer, Grunwald, R, Wang, Larsson, Koolen, Ruf, Howard, etc. (last 5 years)

Define the SR e-detector as  $M_n^{SR} := \sum_i \Lambda_n^{(j)}$ , where  $\Lambda^{(j)}$  is a  $\mathscr{P}% _{j}$  -e-process started at time  $j.$ *n* ∑ *j*=1  $\Lambda_n^{(j)}$  $\Lambda^{(j)}$  is a  $\mathscr{P}% _{j}$  -e-process started at time  $j$ 

#### **E-detector**

 $\mathscr{F} \equiv (\mathscr{F}_n)_{n>1}$ : filtration, and  $\mathscr{F}_n$  need not be  $\sigma(X_1, ..., X_n)$ .  $\mathscr P$ : pre-change set of distributions over infinite sequence

An e-detector for  $\mathscr P$  is a nonnegative process  $M \equiv (M_n)_{n>1}$ such that  $\mathbb{E}_p[M_\tau] \leq \mathbb{E}_p[\tau]$  for all  $P \in \mathscr{P}, \tau \in \mathscr{T}$ .

 $\mathscr{T}$ : set of all stopping times with respect to  $\mathscr{F}$ 

## **Nonasymptotic ARL control**

Define stopping time *N*\*  $J_{1/\alpha}^* := \inf\{t \geq 1 : M_t \geq 1/\alpha\}$ .

> Theorem: If  $M$  is an e-detector for  $\mathscr{P}$ , then the ARL is  $\mathbb{E}_P[N^*_{1/\alpha}]$ ] ≥ 1/*α* ∀*P* ∈

# *Computationally efficient* **e-detectors using "baseline" e-processes**

Often, our e-process can be written as 
$$
\Lambda_n^{(j)} := \prod_{i=j}^n L_i
$$
  
where  $\mathbb{E}_p[L_n | \mathcal{F}_{n-1}] \leq 1$  for all  $P \in \mathcal{P}$ .

Then,  $M_n^{SR} := L_n(M_{n-1}^{SR} + 1)$  can be computed online, like the Shiryaev-Roberts procedure.

Example: likelihood ratios, if  $\mathscr{P} = \{P\}$  is a point null, but also any setting with a nonnegative supermartingale for composite  $\mathscr{P}$ .

# **Cleveland Cavaliers (NBA, 2011-18)**

#### Plus-minus = Points scored - Points conceded

**Plus-Minus of the Cavaliers** 



**Game Date** 

Is there a changepoint from a negative to a positive plus-minus?

E-detector: a nonparametric framework to answer such questions.

The paper: related work (Lorden, Pollak, Siegmund, Tartakovsky, Veeravalli, Xie, Harchaoui, and many others)

# **Why is it hard?**

Challenges: the plus-minus stat of a game is a bounded r.v. between  $[-100, +100]$  (let's rescale to  $[0,1]$ ) and its mean varies over time (with form, injuries, etc), and we know nothing else about the distribution.

 $X_1, X_2, ... \sim P$  for some  $P \in$ 

where  $\mathscr{P} := \{ P \text{ on } [0,1]^{\infty} : \mathbb{E}[X_n] \text{ past} ] \leq 1/2 \text{ for all } n \geq 1 \}$ 

If there is a change at time  $\nu$ , then  $X_{\nu}, X_{\nu+1}, \ldots \sim \mathcal{Q}$  for some  $\mathcal{Q} \in$ where  $\mathcal{Q} := \{ Q \text{ on } [0,1]^{\infty} : \mathbb{E}[X_n] \text{ past} ] > 1/2 \text{ for all } n \geq 1 \}$ 

Here  $P, Q$  are distributions on *infinite* sequences of observations, so the data are not iid (neither i nor id), and are only restricted by their conditional means

# **Method**

# $X_1, X_2, ... \sim P$  for some  $P \in$ where  $\mathscr{P} := \{ P \text{ on } [0,1]^{\infty} : \mathbb{E}[X_n] \text{ past} ] \leq 1/2 \text{ for all } n \geq 1 \}$

If there is a change at time  $\nu$ , then  $X_{\nu}, X_{\nu+1}, \ldots \sim \mathcal{Q}$  for some  $\mathcal{Q} \in$ where  $\mathcal{Q} := \{ Q \text{ on } [0,1]^{\infty} : \mathbb{E}[X_n] \text{ past} ] > 1/2 \text{ for all } n \geq 1 \}$ 

Fix  $Q$  for now: we have seen that we can mix over  $Q$ , or plug-in.

#### For a fixed  $Q$ , recall the previous two lectures: *t* ∏ *i*=1  $(1 + \lambda_Q^*(X_t - 1/2))$

Is the optimal supermartingale. So plug this into the e-detector.

# **Result**

#### CP detected at 2015-03-10



Our e-detector announces a change point during the 2014-15 season. It controls the average run length (ARL) at  $1/\alpha = 1000$ , which is more than twelve seasons of games.

More broadly: the e-detector can help detect changes when the pre-change and post-change distributions are composite, non-stationary and nonparametrically specified (eg: no common reference measure, no likelihoods, infinite dimensional nuisance parameters, etc.)

(Simulation, asymptotics are intractable for ARL.)

# **A short list of e-processes for other settings**

• ANY  $\mathscr P$  for which max-likelihood (after smoothing or profiling) is feasible (eg: shape constraints like monotone, log-concave)

> Wasserman, Balakrishnan, Ramdas'20 Universal inference

•  $\mathscr{P} = \{P : X_1, X_2, \ldots \}$  are exchangeable },  $\mathscr{Q}$  unrestricted

Vovk'21 (Testing randomness online) Ramdas, Ruf, Larsson, Koolen'21 (Testing exchangeability: fork-convex hulls, supermartingales and e-processes)

•  $\mathscr{P} := \{ (P \times P)^{\infty} : \text{for any } P \}, \mathscr{Q} := \{ (P \times Q)^{\infty} : \text{for any } P \neq Q \}$ 

Shekhar, Ramdas'21 (Nonparametric two-sample testing by betting)

## **Some more e-processes for other settings**

$$
\mathcal{P} := \{ P^{\infty} : \mathbb{E}[X] \le \mu, \text{Var}(P) \le \sigma^2 \}
$$
  

$$
\mathcal{Q} := \{ P^{\infty} : \mathbb{E}[X] > \mu, \text{Var}(P) \le \sigma^2 \}
$$

Wang, Ramdas'23 (Catoni-style CSs for heavy-tailed mean estimation)

•  $\mathscr{P} := \{P^\infty : \mathbb{E}[X] \leq \mu, \text{Var}(P) \leq \sigma^2, \text{adversary corrupts } \epsilon \text{ fraction of data}\}\$ :=  $\{P^{\infty}: \mathbb{E}[X] > \mu, \text{Var}(P) \leq \sigma^2, \text{adversary corrupts } \epsilon \text{ fraction of data}\}$ 

Wang, Ramdas'23 (Huber-robust confidence sequences)

• 
$$
\mathscr{P} := \{P_{XY}^{\infty} : \text{where } P_{XY} = P_X \times P_Y\},\
$$
  
 $\mathscr{Q} := \{P_{XY}^{\infty} : \text{where } P_{XY} \neq P_X \times P_Y\}$ 

Shekhar, Ramdas'23 and Podkopaev et al.' 23 (Nonparametric independence testing by betting)

# **Detection delay: suppose Q is known**

For a changepoint stopping rule  $N$ , Define worst detection delay  $(N) := \sup \mathbb{E}_{P,\nu,Q}[(N-\nu)_+|N>\nu]$ *ν*≥0

If the baseline increment  $L_n$  is function of only  $X_n$ and the post-change observations from  $Q$  are strongly stationary, then  $\mathscr{D}(N^{SR}) \leq \frac{\log(1/\alpha)}{\sqrt{n}} + \frac{\mathbb{E}_{Q} \log^2 L_1}{\sqrt{n}} + 1$  $\varrho \log L_1$  $+\frac{\mathbb{E}_Q \log^2 L_1}{\sqrt{\mathbb{E}_Q \log^2 L_1}}$  $\frac{2}{(E_Q \log L_1)^2} + 1$ 

where  $N^{SR} := \inf\{t \geq 1 : M_t^{SR} \geq 1/\alpha\}.$ 

Minimizing delay corresponds to maximizing  $K:=\mathbb{E}_{Q}\log L_{1}$ , but we usually don't know *Q* .

Let  $(L^\lambda)_{\lambda \in \Pi}$  be a family of baseline processes indexed by  $\lambda$  .  $\lambda^* := \arg \max_{\lambda \in \Pi} \mathbb{E}_{\mathcal{Q}} \log L_1^{\lambda}$  depends on  $\mathcal{Q}.$ 

# **Mixtures of e-detectors**

A convex combination (mixture) of e-detectors is also an e-detector. (ARL control maintained)

This fact can be used to take statistically and computationally efficient mixtures over  $\Pi$  (implicitly over  $\mathcal Q$ ), in order to derive e-detectors that adapt to unknown  $\mathcal Q.$ 

We derive e-detectors where the number of mixture components increases (logarithmically) with time.

#### **Detection delay for the mixture e-SR procedure**

$$
\mathcal{D}(N^{mSR}) \le \frac{g_{\alpha}}{D(Q||\mathcal{P})} + \frac{\mathbb{V}_{Q}[\log L_{1}^{\lambda^{*}}]}{D^{2}(Q||\mathcal{P})} + 1
$$

Defining  $\Delta^* = \nabla \psi(\lambda^*)$  as the "signal strength", lying in  $[\Delta_L, \Delta_U]$ , we have *g<sup>α</sup>* ≤ inf  $\inf_{\eta>1}\eta$   $\left[\log(1/\alpha)+\log(1+\lceil\log_\eta$  $\psi^*(\Delta_U)$  $\left[\frac{\varphi^{*}(-U)}{\psi^{*}(\Delta_L)}\right]$ 

Qualitatively similar results in the non-separated case.

#### **Takeaway messages**

A  $\mathscr{P}$ -e-detector is a nonnegative adapted process  $M \equiv (M_n)_{n>1}$ such that  $\mathbb{E}_p[M_{\tau}] \leq \mathbb{E}_p[\tau]$  for all  $P \in \mathscr{P}, \tau \in \mathscr{T}$ .

A  $\mathscr{P}$ -e-process is a nonnegative adapted process  $\Lambda \equiv (\Lambda_n)_{n>1}$ such that  $\mathbb{E}_P[\Lambda_\tau] \leq 1$  for all  $P \in \mathscr{P}, \tau \in \mathscr{T}$ .

Eg: the SR e-detector as 
$$
M_n^{SR} := \sum_{j=1}^n \Lambda_n^{(j)}
$$
,  
where  $\Lambda^{(j)}$  is a *P*-e-process started at time *j*.

Thresholding at  $1/\alpha$  controls ARL at  $1/\alpha$ . Mixtures of e-detectors are also e-detectors. Baseline e-processes (eg: exponential) enable online computation.

Now, we can perform changepoint detection in a slew of nonparametric settings, sometimes good bounds on detection delay.

# **Today's talk**

Reducing partitioned change detection to sequential testing

- 2. Reducing non-partitioned change detection to sequential estimation
- 3. High level summary of game-theoretic statistics

# E-detectors (NEJSDS'23)

When pre- and post-change distributions are known to lie in separate classes (eg: mean change from  $\langle 0 \rangle$  to  $> 0$ )

# Confidence sequences arXiv'23

When pre- and post-change distributions are simply two different distributions in one class (eg: the mean changed from something to something else)

# Via confidence sequences (arXiv)





Shubhanshu Shekhar (CMU)

TLDR: Reducing sequential (nonparametric) change detection to (modern) sequential estimation

#### Sequential Changepoint Detection

Stream of independent  $X$ -valued observations:  $X_1, X_2, \ldots$ 

- For some  $T \in \mathbb{N} \cup \{\infty\}$ :
	- $\triangleright$  X<sub>t</sub> ∼ P<sub>0</sub> for  $t < T$
	- $\triangleright$   $X_t \sim P_1 \neq P_0$  for  $t > T$
- $\blacktriangleright$  Mild requirements on the distributions:
	- $\blacktriangleright$  Both  $P_0, P_1$  are unknown
	- $\blacktriangleright$   $P_0, P_1 \in \mathcal{P}$  for some known class of distributions  $\mathcal{P}$
- Decide between

$$
H_0: T = \infty, \quad \text{versus} \quad H_1: T < \infty.
$$

- **Objective:** Define a stopping time  $\tau$  to declare a detection, that
	- $\triangleright$  minimizes false alarms under  $H_0$
	- $\blacktriangleright$  has a small detection delay,  $(\tau \mathcal{T})^+$ , under  $H_1$

A "confidence sequence (CS)" for a parameter *θ* is a sequence of confidence intervals (*Ln*, *Un*) that are constructed from the first  $n$  samples, and have a uniform (simultaneous) coverage guarantee.  $\mathbb{P}(\forall n \geq 1 : \theta \in (L_n, U_n)) \geq 1 - \alpha$ .

Equivalently, for any stopping time  $\tau : \mathbb{P}(\theta \in (L_{\tau}, U_{\tau})) \geq 1 - \alpha$ .

Darling, Robbins '67, '70s Lai '76, '84 Robbins, Siegmund '70s

Much stronger than the pointwise (fixed-sample) confidence interval (CI) guarantee:

$$
\forall n \geq 1, \mathbb{P}(\theta \in (\tilde{L}_n, \tilde{U}_n)) \geq 1 - \alpha.
$$

#### Overview of our results

If we can construct<br>a CS for  $\theta$ 

If we can construct  $\begin{array}{c} \mathbf{a} \subset \mathbf{S} \text{ for } \theta \end{array}$  We can detect changes in  $\theta$ changes in  $\theta$ 

 $\sim$  durations at over the of  $\epsilon$  at  $\sim$ Our Reduction: at every time *t*, start a *new* level-α confidence We declare a change when their intersection is empty. sequence based on  $X_t, X_{t+1}, \ldots$  (there are t active CSs after t steps).

 $K$ ey theorem 1: This method controls ARL (FFA) at level  $1/\alpha$ . ! tight control over the expected detection delay

problems for which the confidence sequence shrinks like  $\widetilde{O}(1/\sqrt{t})$ . Key theorem 2: The detection delay is minimax optimal for many

## Main Assumptions

- $\triangleright$  We work with distribution class  $\mathcal{P} = \{P_{\theta} : \theta \in \Theta\}.$
- **Possibly infinite dimensional**  $\Theta$  **endowed with metric d.**
- $P_0 = P_{\theta_0}$  and  $P_1 = P_{\theta_1}$  for  $\theta_0, \theta_1$  such that  $d(\theta_0, \theta_1) > 0$ .

#### Assumptions

1. Uniformly decaying width: We can construct a CS  ${C_t(\theta): t \geq 1}$  for all  $\theta \in \Theta$ , satisfying

> $\sup_{\theta \in \Theta} \sup_{\theta \in \Theta} d(\theta', \theta'') \leq w_t \equiv w_t(\Theta, \alpha),$  $\theta \in \Theta$   $\theta', \theta'' \in C_t(\theta)$

such that  $\lim_{t\to\infty} w_t = 0$ .

2. Enough pre-change data: Under  $H_1$ , the changepoint T is large enough to ensure  $w_{\mathcal{T}} < \Delta := d(\theta_1, \theta_0)$ .

 $\blacktriangleright$  Control over the detection delay under  $H_1$ :

Introduce the "good event":  $\mathcal{E} = \{ \forall t \leq T : \theta_0 \in C_t \}$ 

If 
$$
w_t = \mathcal{O}\left(\sqrt{\log \log t/t}\right)
$$
, then  
\n
$$
\mathbb{E}[(\tau - T)^+ | \mathcal{E}] = \mathcal{O}\left(\frac{\log \log(1/\Delta)}{\Delta^2}\right) \text{ where } \Delta = d(\theta_0, \theta_1).
$$

• Can generalize to arbitrary  $w_t \to 0$  (next slide).

## Detection Delay Analysis: general  $w_t$



## Detection Delay Analysis: general  $w_t$



#### Detection Delay Analysis: general  $w_t$



## Applications

 $\blacktriangleright$  Mean-shift detection with univariate Gaussians.  $\blacktriangleright$   $P_{\theta_0} = \mathcal{N}(\theta_0, 1)$ , and  $P_{\theta_1} = \mathcal{N}(\theta_1, 1)$ , with  $\Delta = |\theta_1 - \theta_0|$ .

- **In Mean-shift detection with bounded observations** 
	- $\triangleright$   $P_{\theta_0}$  and  $P_{\theta_1}$ , supported on [0, 1] with  $\Delta = |\theta_1 \theta_0|$ .
- $\triangleright$  Changes in CDF
	- $\blacktriangleright$   $\Delta = d_{\mathcal{KS}}(\theta_0, \theta_1)$ , with  $\theta_i = \text{CDF of } P_{\theta_i}$ .
- $\blacktriangleright$  Two-sample changepoint detection
	- $\blacktriangleright$   $\theta_0 = P \times P$ ,  $\theta_1 = P \times Q$ , and  $\Delta = \text{MMD}(P, Q)$ .
- ! Several other tasks: distribution shifts in ML, nonparametric regression, exponential family.

# Summary

# Using "E-detectors"

#### arXiv: 2203.03532 (under review)

When pre- and post-change distributions are known to lie in separate classes  $\mathscr{P},$ (eg: mean change from  $\langle 0$  to  $> 0$ )

Key idea: reduce change detection to repeated sequential testing of  $\mathscr P$  vs.  $\mathscr Q$ 

Method: at every time t, start a new level- $\alpha$  sequential test based on  $X_{t}, X_{t+1}, \ldots$  and declare change when accumulated evidence crosses 1/*α*

#### Main theorem:  $ARL \geq 1/\alpha$  Main theorem:  $ARL \geq 1/\alpha$

# Using "confidence sequences"

ICML 2023 + to-be-arXived

When pre- and post-change distributions are unknown but from the same class (eg: the mean changed from *a* to *b*)

Key idea: reduce change detection to repeated sequential estimation

Method: at every time t, start a new level-α confidence sequence based on  $X_{t}, X_{t+1}, \ldots$  and declare change when their intersection is empty.

Advantage: sequential testing/estimation are basic problems, increasingly well studied.

# **Today's talk**

Reducing partitioned change detection to sequential testing Reducing non-partitioned change detection to sequential estimation

3. High level summary of game-theoretic statistics

#### What is game-theoretic statistics?

A subfield whose quest is to quantify uncertainty in statistical inference tasks like

- **— hypothesis testing**
- **estimation (confidence sets)**
- **probabilistic forecasting**
- **— model selection**
- **change detection**

by using game-theoretic intuition, language and formalism.

Helps design **new nonparametric inference procedures** which

- Have strong *theoretical guarantees under weaker assumptions*,
- Provide *greater flexibility in applications*,
- Combine the use of *prior knowledge with frequentist guarantees*,
- Are designed to be adaptive to the underlying data.

# The 4 characters in SAVI



Understanding their properties and designing "good" ones has been the focus of all my research for many years.

#### **E-processes are nonparametric, composite generalizations of likelihood ratios**

Every likelihood ratio is a test martingale. *Further, every test martingale is (implicitly) a likelihood ratio.*

If  $(M_t)$  is a test martingale for  $\mathscr{P}$ , then for every  $P \in \mathscr{P}$ , we can write  $M_t = \frac{2(t-1)}{P(Y)}$  for some  $Q$  — it is a "simultaneous likelihood  $Q(X_1, ..., X_t)$  $\frac{P(X_1, \ldots, X_t)}{P(X_1, \ldots, X_t)}$  for some  $Q$ 

ratio", despite there being no dominating reference measure for  $\mathscr{P}$ .

Test martingales are at the *heart* of parametric inference, (likelihood ratio) — but also nonparametric inference.

E-processes generalize test martingales: *they exist more generally*.

Ramdas, Ruf, Larsson, Koolen (arXiv:2009.03167)

\* : no dominating reference measure for the set of distributions

#### **Nontrivial (e-power) test martingales exist**

Two-sample testing\* Bounded means\* Exchangeability\* (in shrunk filtration) Independence testing\* (in shrunk filtration) T-test (in shrunk filtration) Testing symmetry\*

#### **Nontrivial test supermartingales exist**

SubGaussian distributions\* (or any bounded MGF)

Robust, heavy-tailed mean estimation\*

#### **Nontrivial test martingales exist**

Two-sample testing\* Bounded means\* Exchangeability\* (in shrunk filtration) Independence testing\* (in shrunk filtration) T-test (in shrunk filtration) Testing symmetry\*

#### **Nontrivial e-processes exist**

Any composite  $\mathscr P$  : "universal inference" Exchangeability\* (in original filtration) T-test (in original filtration)

#### **Nontrivial test supermartingales exist**

SubGaussian distributions\* (or any bounded MGF)

Robust, heavy-tailed mean estimation\*

#### **Nontrivial test martingales exist**

Testing symmetry\* Two-sample testing\* Bounded means\* Exchangeability\* (in shrunk filtration) Independence testing\* (in shrunk filtration) T-test (in shrunk filtration)

#### **Some current and future directions**

- 1. For a new (nonparametric) problem, how do we *design* the game and *learn* to bet?)
- 2. When do *nontrivial* test martingales (not) exist? When do *nontrivial* test supermartingales (not) exist? When do *nontrivial* e-processes (not) exist?
- 3. How do we move beyond this tutorial's topics to, say, game-theoretic model selection?
- 4. Everything today was nonasymptotic. What about anytime-valid asymptotics? (Few arXiv preprints)
- 5. How do we tie together game-theoretic statistics with game-theoretic probability?



Peter

Grunwald



Glenn

Shafer



Volodya

Vovk





Ruf



Koolen



Martin Larsson Johanna Ziegel









Ruodu







Ian Howard Balsubramani Waudby-Smith Akshay Steve













Justin **Whitehouse** Jaehyeok Shin



Neil

Xu



![](_page_44_Picture_25.jpeg)

![](_page_44_Picture_26.jpeg)

Ben

**Chugg** 

![](_page_44_Picture_27.jpeg)

![](_page_44_Picture_28.jpeg)

![](_page_44_Picture_29.jpeg)

Duan

Two general frameworks for sequential change detection with composite, nonparametric classes

![](_page_45_Picture_1.jpeg)

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