Efficient Private Mean Estimation

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Georgia Tech AI4OPT Seminar Series

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With Samuel B. Hopkins (MIT) and Mahbod Majid (University of Waterloo MMath → CMU PhD)

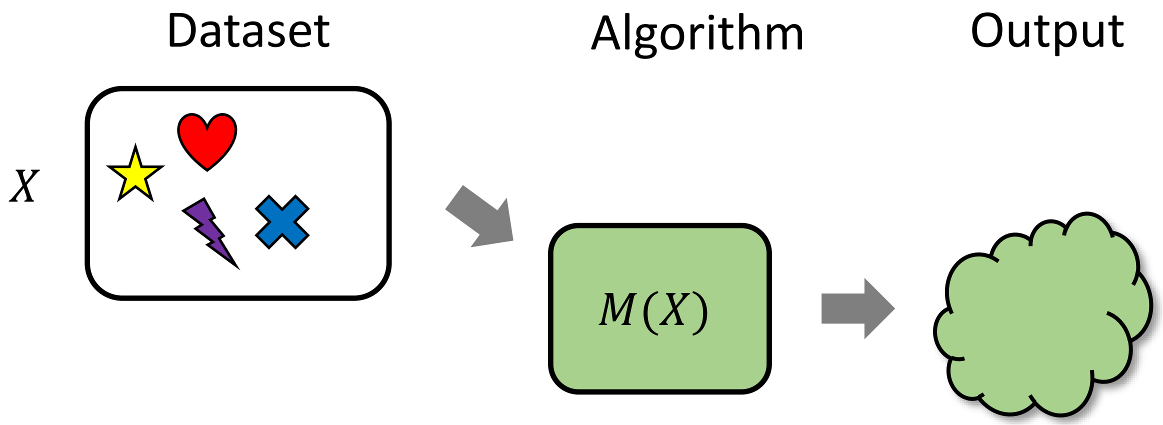
A Sneak Preview

- An all-in-one private mean estimator!
- Efficient, private, with near-optimal sample complexity
- But also robust with sub-Gaussian rates
- More broadly:
  - Addresses a fundamental deficiency in our understanding of DP estimation
  - Algorithmic high-dimensional statistics strikes again!
- A new area to explore for multiple communities
  - Robust Statistics
  - Sum-of-Squares

Privacy?

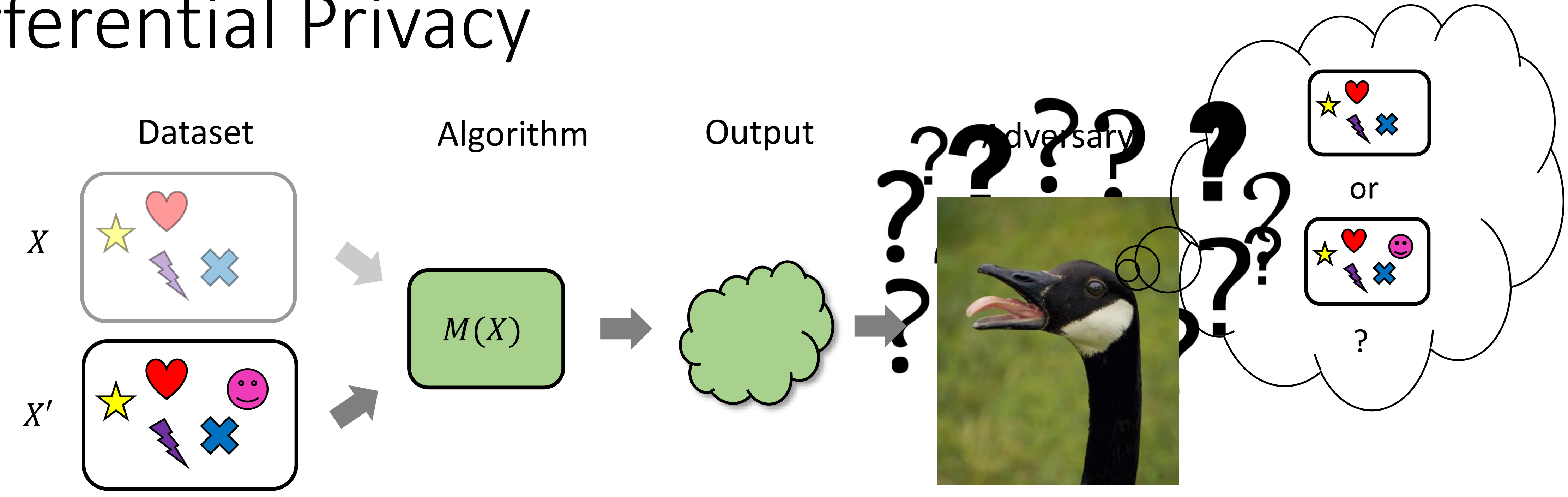
How do we ensure statistics don’t leak information about individual datapoints?

Differential Privacy



[Dwork, McSherry, Nissim, Smith], 2006

Differential Privacy

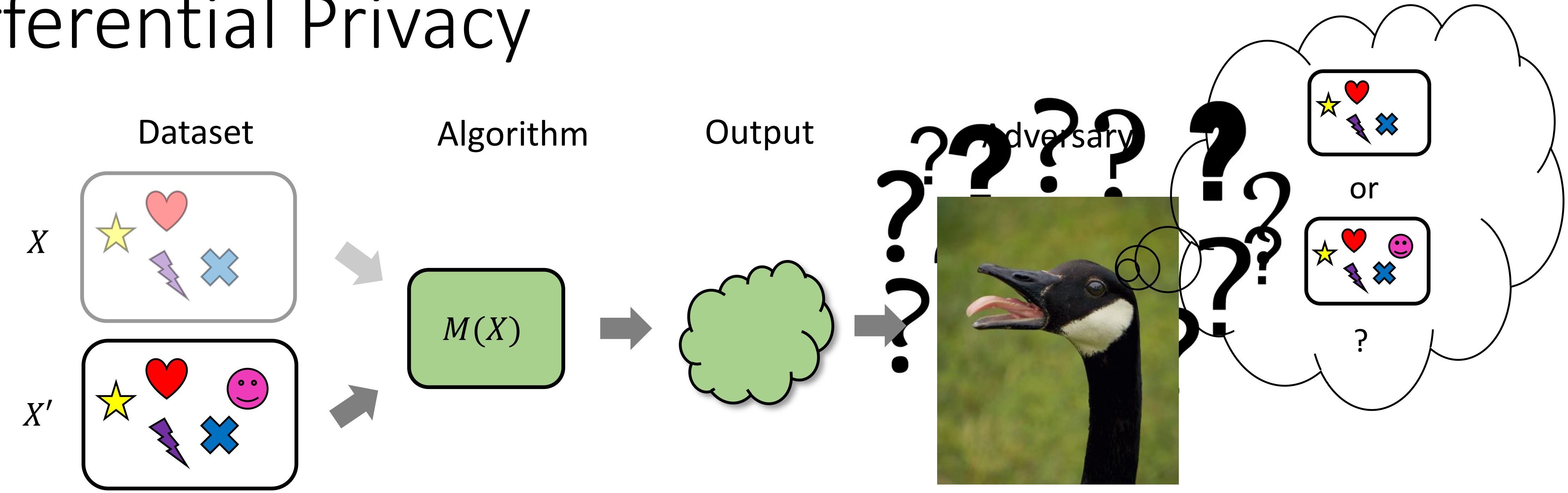


- $M: D^n \to R$ is (ε, δ)-DP if for all inputs X, X' which differ on one entry:

$$\forall S \subseteq R \qquad \Pr[M(X) \in S] \approx_{\varepsilon,\delta} \Pr[M(X') \in S]$$

[Dwork, McSherry, Nissim, Smith], 2006

Differential Privacy



- $M: D^n \to R$ is (ε, δ)-DP if for all inputs X, X' which differ on one entry:

$$\forall S \subseteq R \qquad \Pr[M(X) \in S] \leq e^{\varepsilon} \Pr[M(X') \in S] + \delta$$

[Dwork, McSherry, Nissim, Smith], 2006

(ε, δ)-Differential Privacy

- "Privacy loss random variable" is bounded by ε with probability $1 - \delta$ for all datasets X and X' which differ in a single entry
- Pure differential privacy: $\delta = 0$
- Approximate differential privacy: $\delta > 0$
- Qualitatively different notions
  - Much easier to design algorithms for approx. DP
- Today: Goal is pure DP algorithms

Non-Private Mean Estimation

Given d-dimensional $X_1, \ldots, X_n \sim p$, where p has covariance $\|\Sigma\|_2 \leq 1$, output $\hat{\mu}$ such that $\|\hat{\mu} - E[p]\|_2 \leq \alpha$ with prob. 99%

- Empirical mean: $\hat{\mu} = \frac{1}{n}\sum X_i$
  - $n = O(d/\alpha^2)$ samples suffice non-privately (rate: $\alpha \leq O(\sqrt{d/n})$
- Fancier techniques: … w.p. $\geq 1 - \beta$ using $n = O\left(\frac{d+\log(1/\beta)}{\alpha^2}\right)$ samples
  - Rate: $\alpha \leq O(\sqrt{(d + \log 1/\beta)/n})$

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  - Rate: $\alpha \leq O(\sqrt{(d + \log 1/\beta)/n})$
  - "Sub-Gaussian rates" [Lugosi, Mendelson], 2019, [Hopkins], 2020, …
  - Not really our focus, but we will eventually get it for free
- Today: pretend we start with a "coarse estimate" of $E[p]$
  - i.e., by recentering, $\|E[p]\|_2 \leq O(\sqrt{d})$

Algorithms for Private Mean Estimation

	Sample Complexity	Privacy Notion	Running Time
Empirical Mean	$O(d)$	Not private	Polynomial

The plan from here…

1. Three Deficient Private Algorithms
2. Fixing the Exponential Mechanism
3. The Algorithm
4. The Bigger Picture

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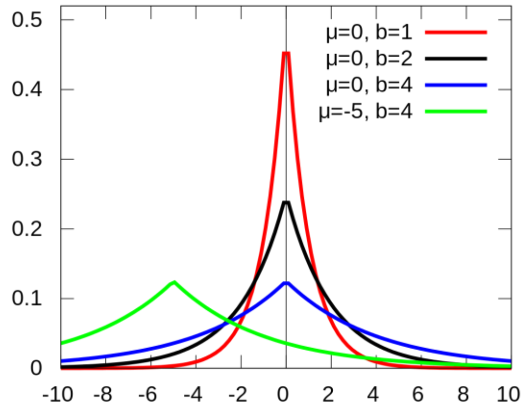
Three Deficient Algorithms

Every flawed algorithm is flawed in its own way…

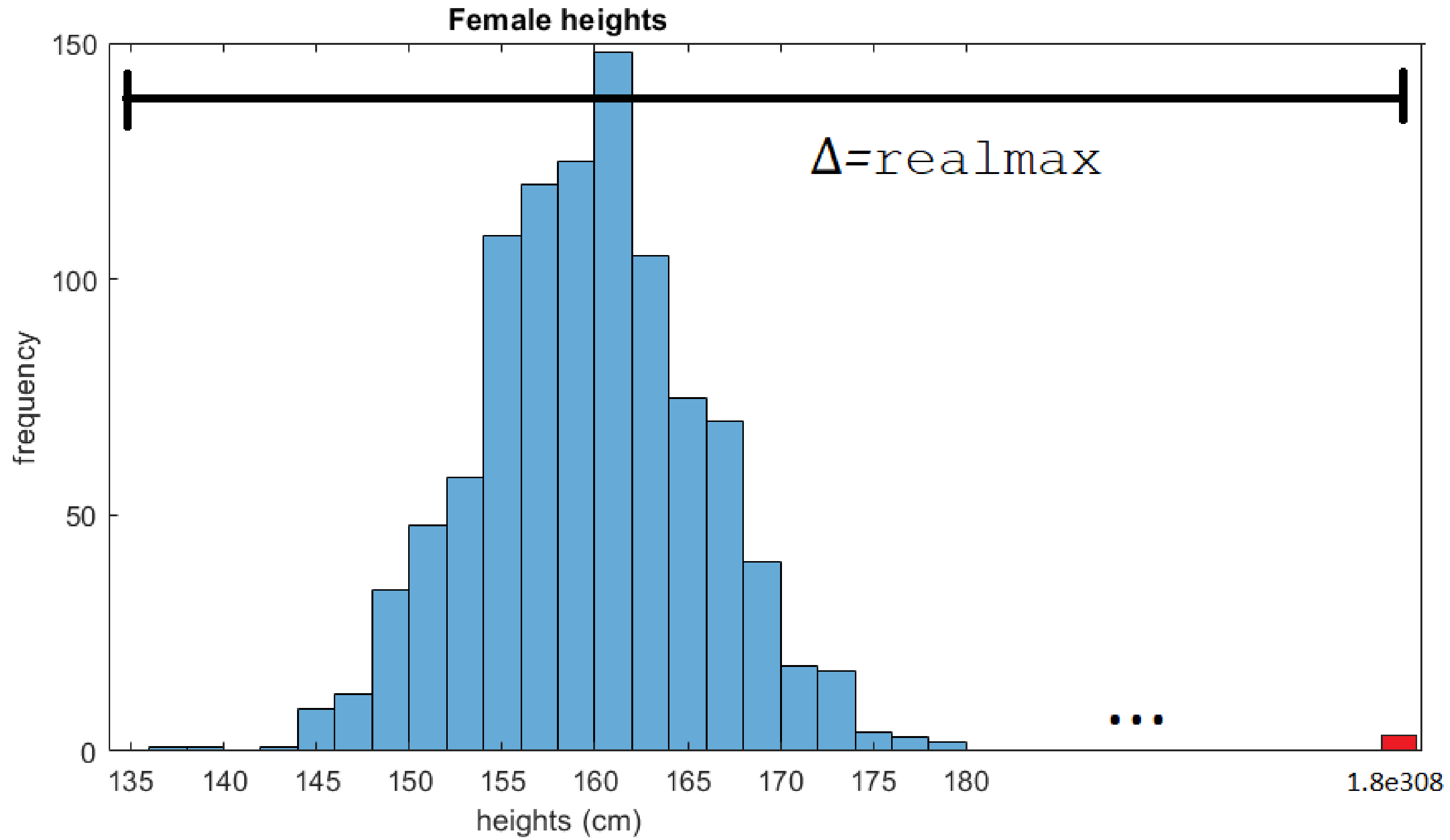
Differential Privacy 101



- Let $f : D^n \to \mathbf{R}^d$ be a vector-valued function of interest
    - e.g., some non-private mean estimation algorithm
- ℓ_1-sensitivity of f: $\Delta_1^f = \max_{X,X':d_H(X,X')=1} \|f(X) - f(X')\|_1$
    - "How much can the function change by modifying one datapoint?"
- The Laplace Mechanism: $f(X) + \mathrm{Lap}\left(\Delta_1^f/\varepsilon\right)^{\otimes d}$ is $(\varepsilon, 0)$-DP
    - "Add Laplace noise to each coordinate, proportional to the ℓ_1-sensitivity"

Laplace Mechanism

- Empirical mean: $f(X) = \frac{1}{n}\sum X_i$
- Sensitivity is infinite



Laplace Mechanism

- Clipped empirical mean: $f(X) = \frac{1}{n}\sum \text{clip}(X_i)$
  - Limit sensitivity by clipping to an ℓ_2-ball of radius $O\left(\sqrt{d}\right)$
  - Biases the statistic, but we can control the bias [K., Singhal, Ullman], 2020
- Resulting ℓ_1-sensitivity: $\Delta_1^f \leq d/n$
- Output $\hat{\mu} = \frac{1}{n}\sum \text{clip}(X_i) + \text{Lap}(d/\varepsilon n)^{\otimes d}$
- ℓ_2 error due to noise $\approx d^{1.5}/\varepsilon n$
- Resulting sample complexity: $O(d^{1.5}/\varepsilon)$

Algorithms for Private Mean Estimation

	Sample Complexity	Privacy Notion	Running Time
Empirical Mean	$O(d)$	Not private	Polynomial
Laplace Mechanism	$O(d^{1.5})$	Pure DP	Polynomial

Gaussian Mechanism (diffs)

- Add Gaussian noise instead of Laplace
- Scaled to ℓ_2-sensitivity instead of ℓ_1-sensitivity
- Resulting sample complexity: $O(d/\varepsilon)$
- Only gives approximate DP $(\delta > 0)$ instead of pure DP $(\delta = 0)$

Algorithms for Private Mean Estimation

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Laplace Mechanism	$O(d^{1.5})$	Pure DP	Polynomial
Gaussian Mechanism	$O(d)$	Approximate DP	Polynomial

Differential Privacy 102

- Privately select an object from a set based on a “score”
- Given: Sensitive dataset $X = X_1, \ldots, X_n$

  Set of objects Q

  Score function $f: D^n \times Q \to \mathbf{R}$
- Output: $q \in Q$ which (approximately) maximizes $f(X, q)$
- Exponential mechanism: Sample q with probability $\propto \exp(\varepsilon \cdot f(X, q))$
- $(\varepsilon, 0)$-differentially private

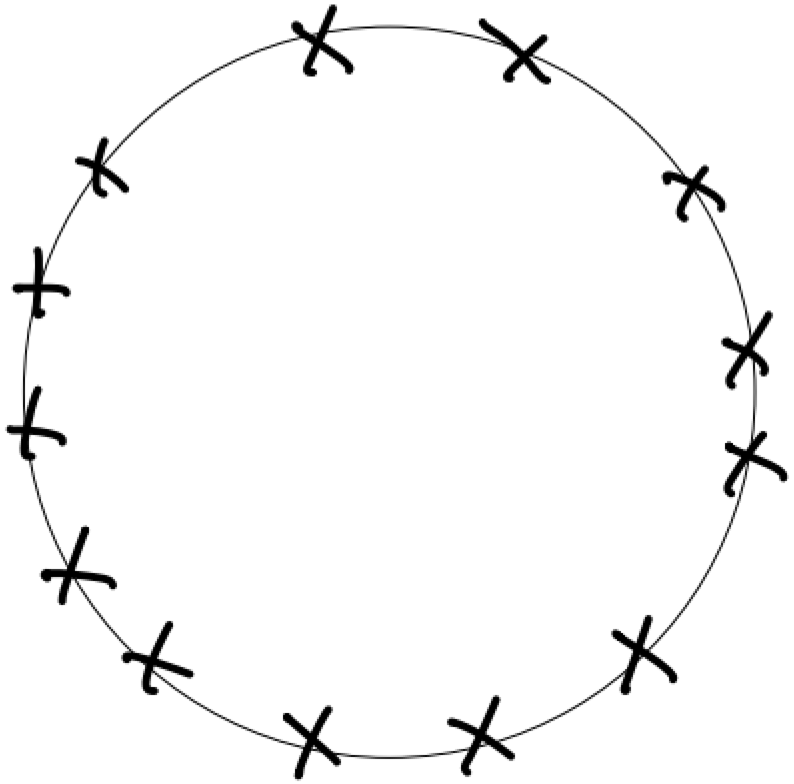
Exponential Mechanism Example

- Running an election
  - Set of objects: election candidates
  - Sensitive dataset: votes
  - Score function: number of votes for each candidate
- Non-privately: pick the highest score
- Privately: sample winner $\propto \exp(\varepsilon \cdot \text{Score})$
- Assign scores, use to noisily pick winner

	15 votes	19 votes	20 votes
$\varepsilon = 0.1$	24% chance	36% chance	40% chance

Exponential Mechanism

- Intuition: Empirical mean should be close to true mean in every 1D projection



[Bun, K., Steinke, Wu], NeurIPS ’19, [K., Singhal, Ullman], COLT ‘20

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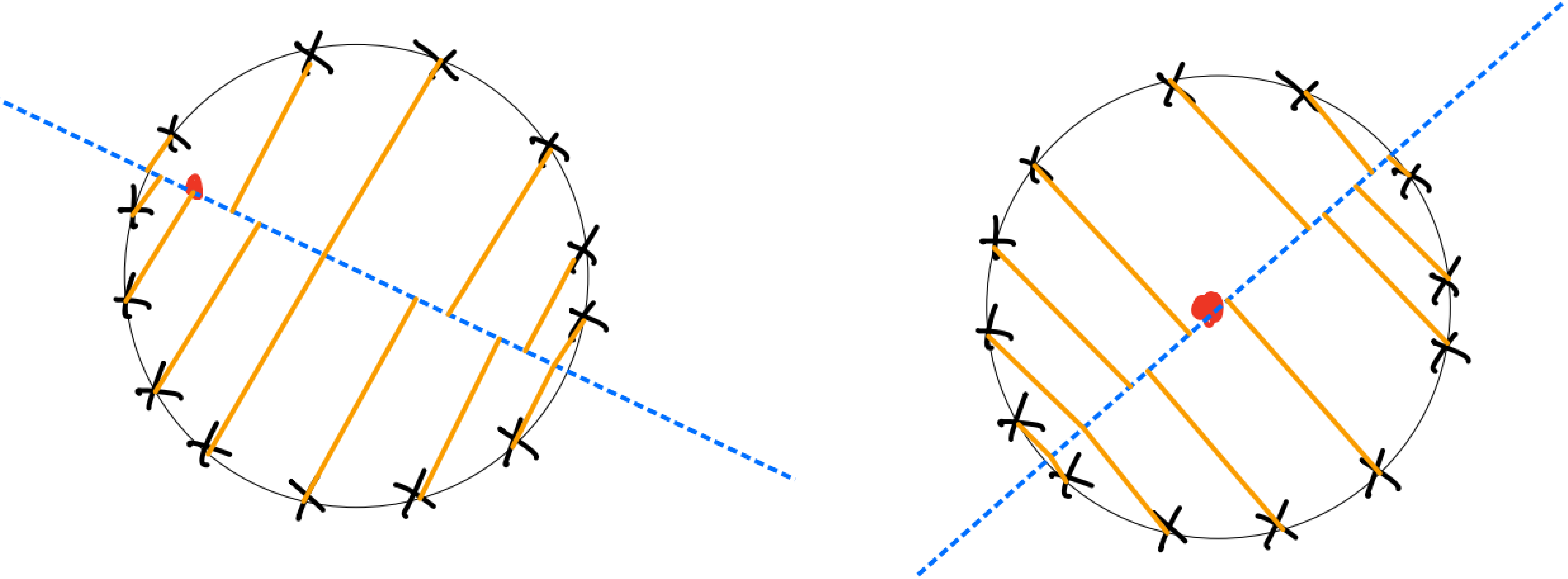

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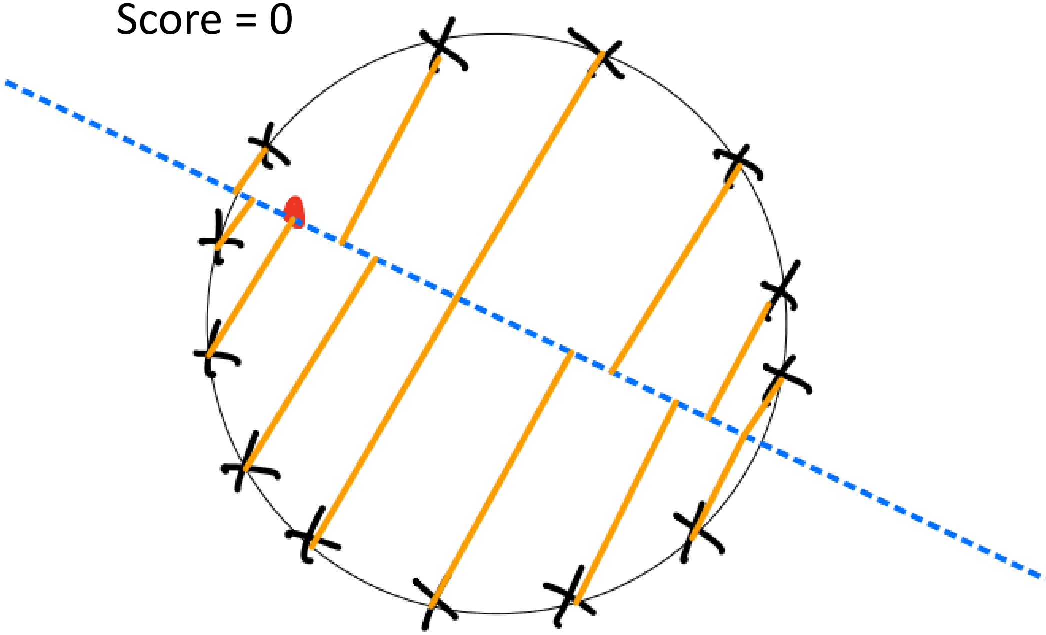
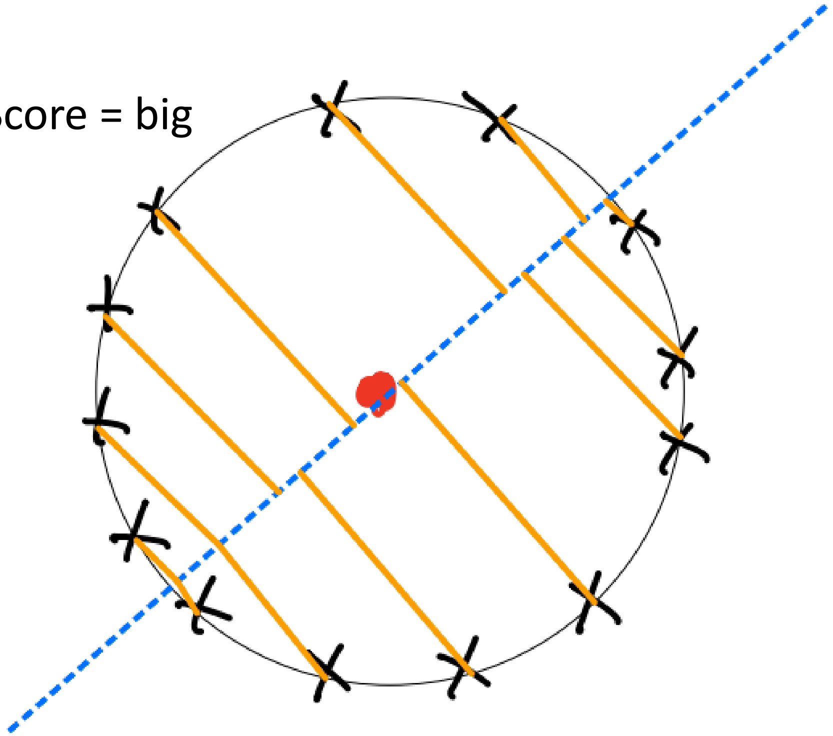
Exponential Mechanism

- Intuition: Empirical mean should be close to true mean in every 1D projection
- Score function at q: "How many points must be changed to have q be far from empirical (clipped) mean in some projection?"

Score = 0



Score = big



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Exponential Mechanism

- Intuition: Empirical mean should be close to true mean in every 1D projection
- Score function at q: "How many points must be changed to have q be far from empirical (clipped) mean in some projection?"
  - Have to look at all projections to compute…
- Set of objects: Cover of set of possible means $\left(2^{\tilde{O}(d)}\right)$
  - Exponentially large…
- Standard analysis gives that $n = \tilde{O}(d)$ samples suffice
- Related: private hypothesis selection via Scheffé's method

[Bun, K., Steinke, Wu], NeurIPS '19, [K., Singhal, Ullman], COLT '20

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Empirical Mean	$O(d)$	Not private	Polynomial
Laplace Mechanism	$O(d^{1.5})$	Pure DP	Polynomial
Gaussian Mechanism	$O(d)$	Approximate DP	Polynomial
Exponential Mechanism	$\tilde{O}(d)$	Pure DP	Exponential

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Exponential Mechanism	$\tilde{O}(d)$	Pure DP	Exponential
Our Algorithm [HKM22]	$\tilde{O}(d)$	Pure DP	Polynomial

Theorem: Given $X_1, \dots, X_n \sim p$ where p has covariance $\|\Sigma\|_2 \leq 1$ and $\|E[p]\|_2 \leq O(\sqrt{d})$, there exists a computationally efficient $(\varepsilon, 0)$-DP algorithm which outputs $\hat{\mu}$ such that $\|\hat{\mu} - E[p]\|_2 \leq \alpha$ with probability $1 - \beta$. It requires $n = \tilde{O}\left(\frac{d+\log(1/\beta)}{\alpha^2 \varepsilon}\right)$ samples.

[Hopkins, K., Majid], STOC 2022

The plan from here…

1. Three Deficient Private Algorithms
2. Fixing the Exponential Mechanism
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Why was the Exponential Mechanism slow?

Two problems to address:

1. Computing the score function for a single candidate is slow
   - Solution: Efficient robust + high-dimensional statistics
2. Have to compute the score function for exponentially many candidates
   - Solution: Efficient log-concave sampling from $\propto \exp\left(\varepsilon \cdot f(X, q)\right)$

1. Efficiently computing the score function

Recent line of work on robust multivariate statistics excels at *efficiently* finding “interesting directions”

- Robust estimation (η-fraction of data is corrupted)
  - [Diakonikolas, K., Kane, Li, Moitra, Stewart], 2016, [Lai, Rao, Vempala], 2016, …
- Sub-Gaussian rates for heavy-tailed estimation (…$+\log(1/\beta)$)
  - [Lugosi, Mendelson], 2019 made efficient by [Hopkins], 2020, …
- Privacy
  - [Hopkins, K., Majid], 2022, [Kothari, Manurangsi, Velingker], 2022, …?

Where do we go now? Finding a direction

- Score function over **directions** (not candidate means)
  - Ideas from [Hopkins], 2020, [Cherapanamjeri, Flammarion, Bartlett], 2019

Definition 6.3 (quadratic optimization problem [Hop20, CFB19]). *Let $Z_1, \ldots, Z_k, \tilde{\mu} \in \mathbb{R}^d$, $r > 0$. Let $\mathrm{QUAD}(\tilde{\mu}, r, Z)$ be the following quadratic program.*

$$\mathrm{QUAD}(\tilde{\mu}, r, Z) :=$$

Z_i: datapoints
$\tilde{\mu}$: current estimate
r: "radius of interest"
v: direction
b_i: 0/1 indicators on points

"Direction v's score: How many points are 'far' in direction v?"

b_i: either 0 or 1

v: unit vector

Either $b_i = 0$
OR
Z_i is far from current estimate $\tilde{\mu}$ in direction v

Where do we go now? Finding a direction

- Score function over **directions** (not candidate means)
  - Ideas from [Hopkins], 2020, [Cherapanamjeri, Flammarion, Bartlett], 2019
- Use SDP relaxation of the quadratic program

Definition 6.4 (SDP Relaxation [Hop20, CFB19]). *Let* $Z_1, \ldots, Z_k, \widetilde{\mu} \in \mathbb{R}^d$, $r > 0$. *Let* $\mathrm{SDP}(\widetilde{\mu}, r, Z)$ *be the following semi-definite program.*

$$\mathrm{SDP}(\widetilde{\mu}, r, Z) := \max_{v,b,B,U,W} \quad \mathrm{Tr}\,(B)$$

$$s.t. \quad \begin{bmatrix} 1 & b^\top & v^\top \\ b & B & W \\ v & W^\top & V \end{bmatrix} \succcurlyeq 0$$

$$B_{ii} = b_i \quad \forall i$$

$$\mathrm{Tr}(V) = 1 \quad \forall i$$

$$B_{ii} \cdot r \leq \langle Z_i - \widetilde{\mu}, W_i \rangle \quad \forall i$$

"Rounding scheme" = just use v

2. Running the Exponential Mechanism efficiently

- Sampling q with probability $\propto \exp\big(\varepsilon \cdot f(X, q)\big)$
  - Naively requires computing $f(X, q)$ for every q in an exponentially-sized cover
- Idea: if $f(X, q)$ is *concave*, then resulting distribution is *log-concave*
- Efficient private samplers for log-concave distributions exist
  - Need multiplicative approximation versus usual total variation guarantee
  - [Bassily, Smith, Thakurta], FOCS 2014, [Mangoubi, Vishnoi], NeurIPS 2022
- Must use continuous version of exponential mechanism
- Additionally need a Lipschitz property

Efficient Log-Concave Sampling

Consider
$$
\begin{aligned}
\text{SDP-VAL}(y;\widetilde{\mu},r,Z) := \max_{v,b,B,W,V} \quad & \operatorname{Tr}(B) \\
s.t. \quad & \begin{bmatrix} 1 & b^\top & v^\top \\ b & B & W \\ v & W^\top & V \end{bmatrix} \succcurlyeq 0 \\
& v_i = y_i \quad \forall i \\
& B_{ii} = b_i \quad \forall i \\
& \operatorname{Tr}(V) = 1 \\
& B_{ii} \cdot r \le \langle Z_i - \widetilde{\mu}, W_i \rangle \quad \forall i
\end{aligned}
$$

Claim: SDP-VAL is both Lipschitz and concave.

Therefore we can sample $\propto \exp(\varepsilon \cdot \mathrm{SDPVAL}(q))$ efficiently.

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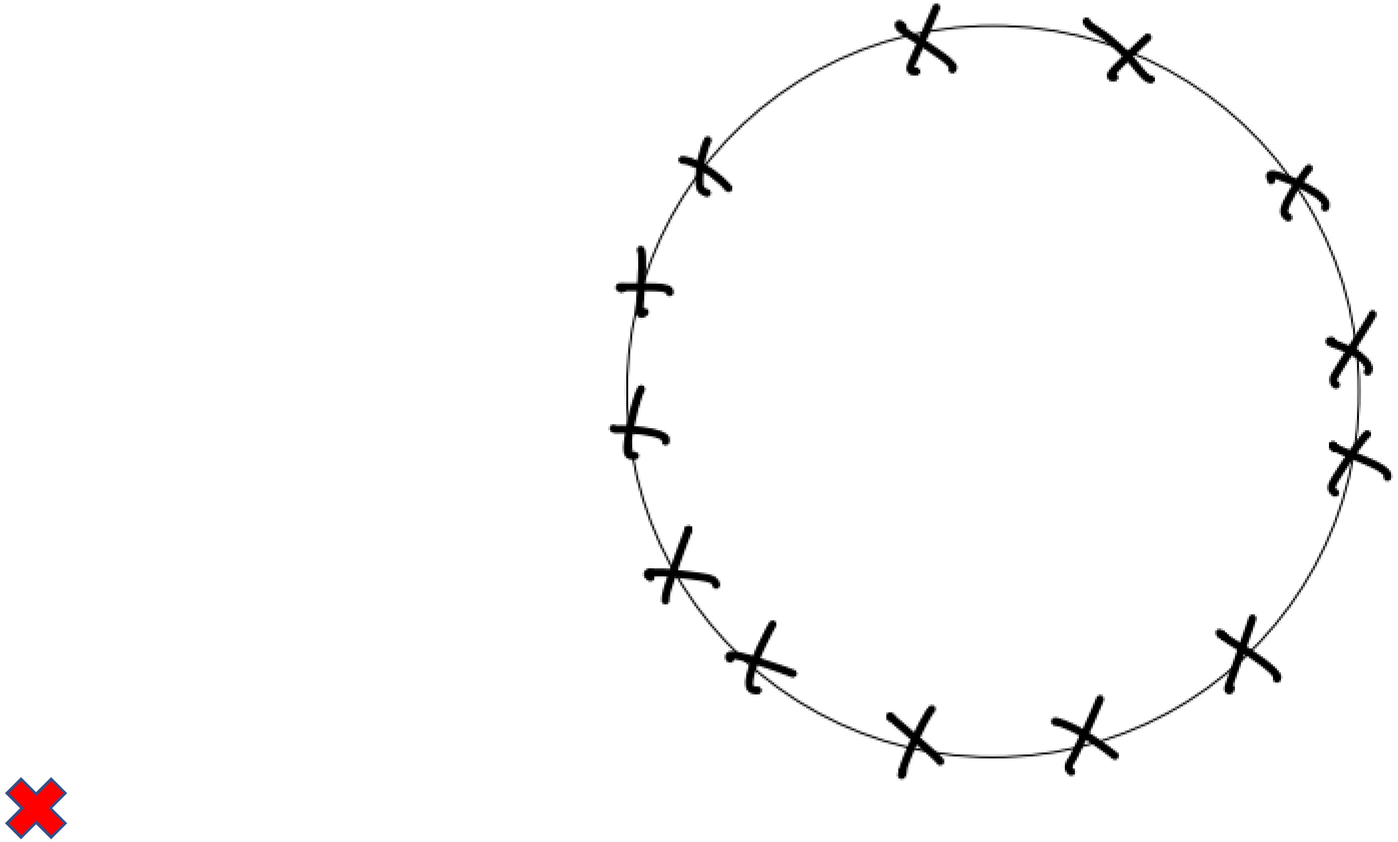
Overall algorithm

1. Find a direction v
   - Use the exponential mechanism with efficient sampling to pick v
     - Via random walk over directions that computes SDP score function at each step
2. Step in that direction
   - Use exponential mechanism to privately estimate the step size
3. Repeat

Run exponential mechanism!



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Running efficient sampler…

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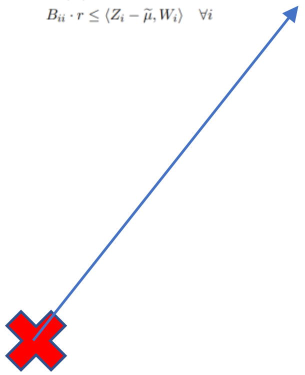

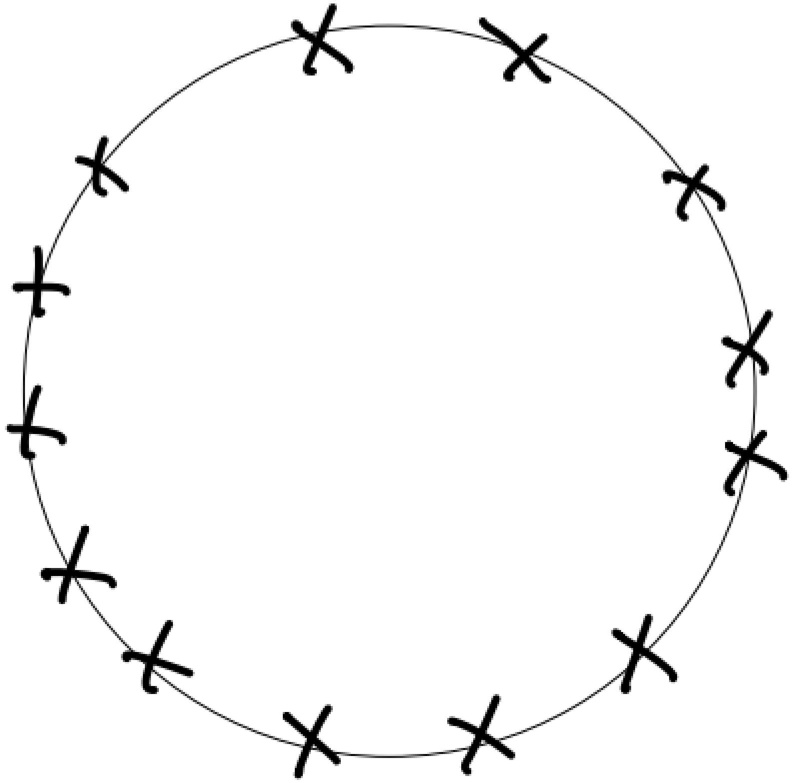

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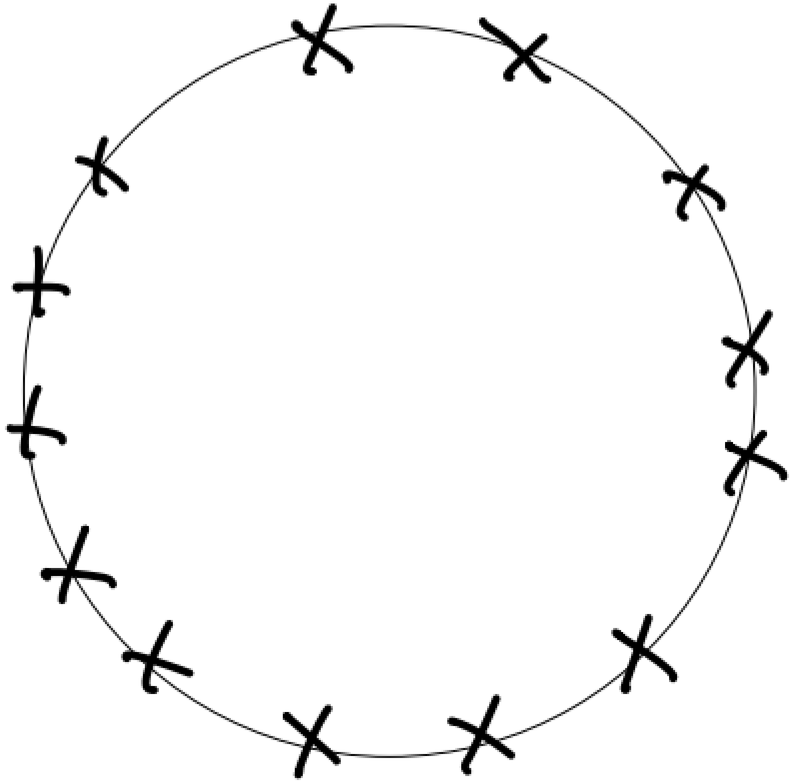
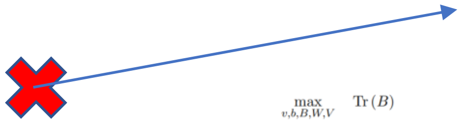
Running efficient sampler…





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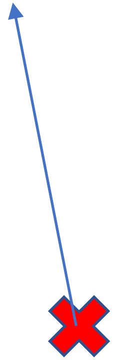

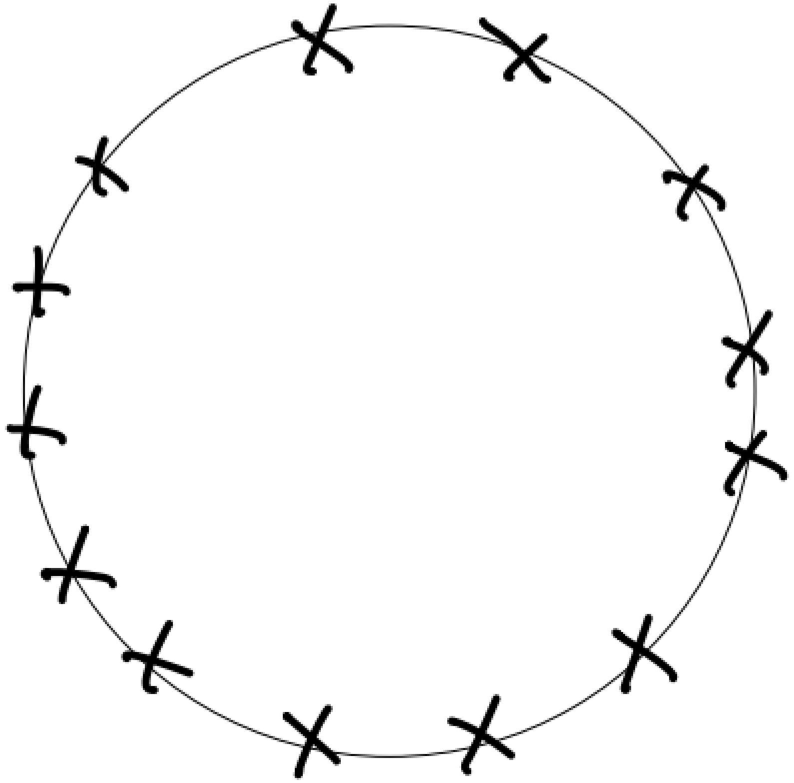

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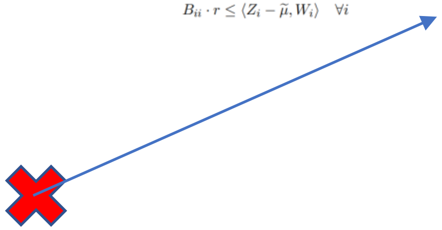
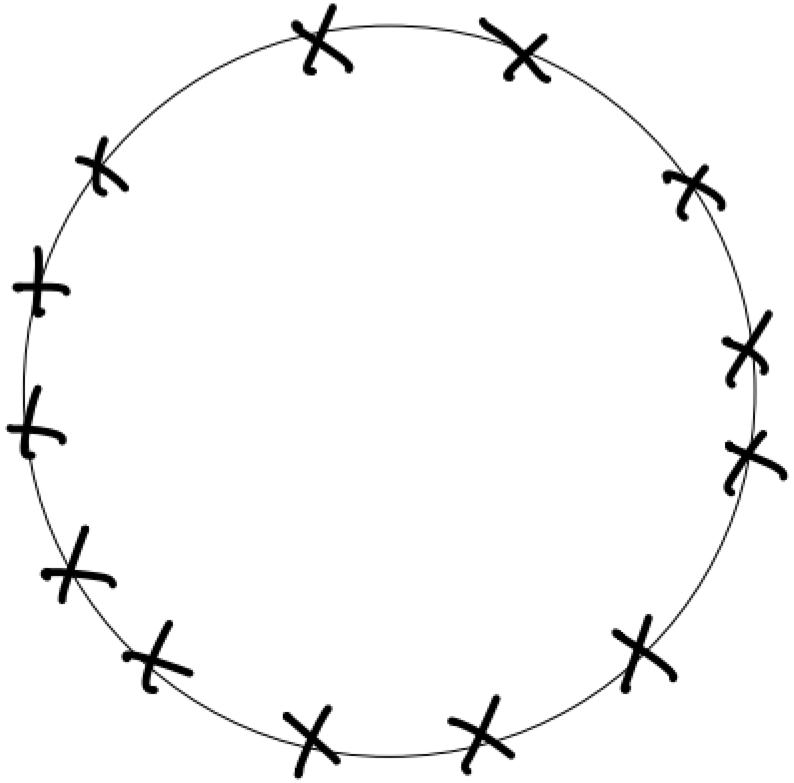
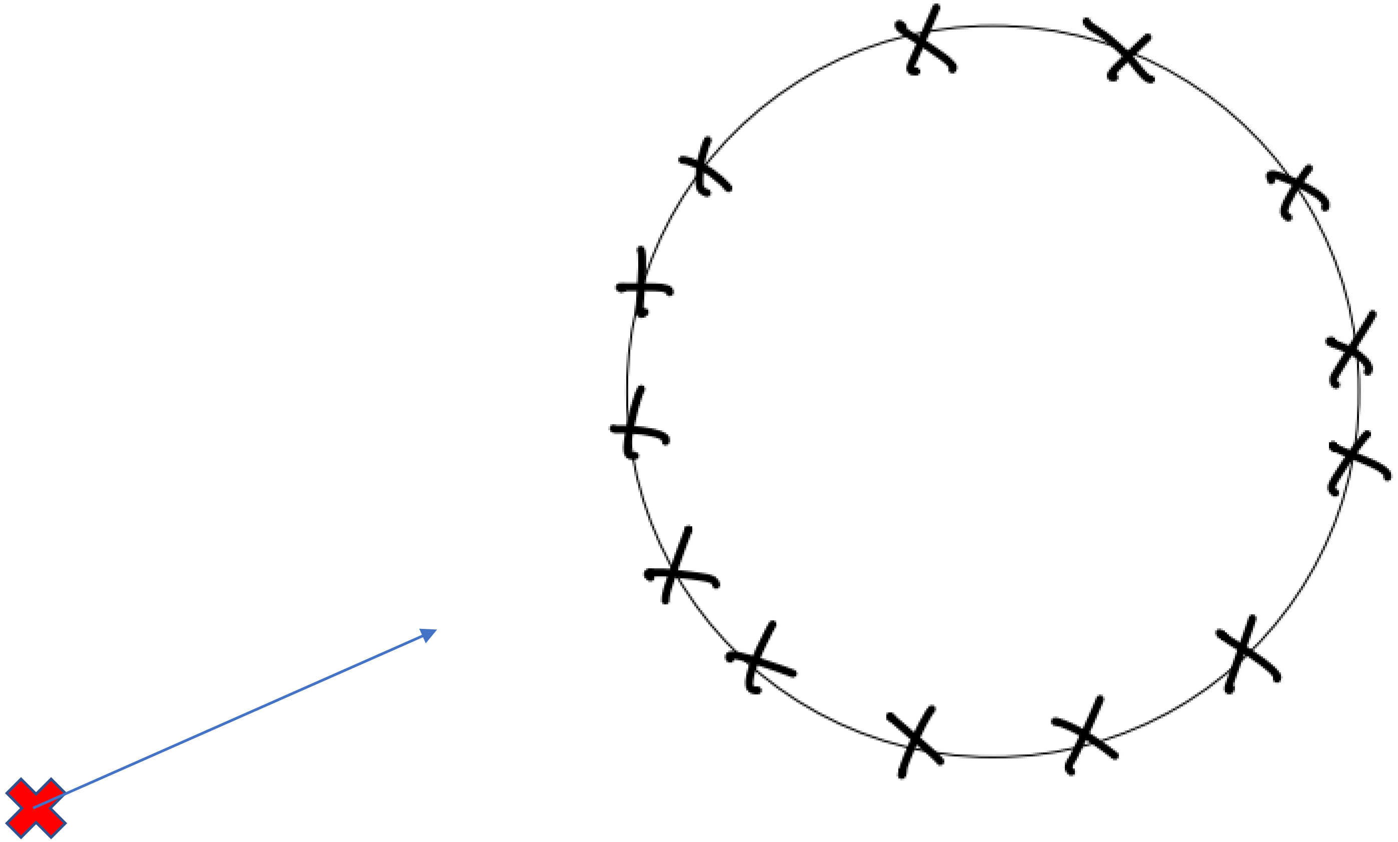




Take a step



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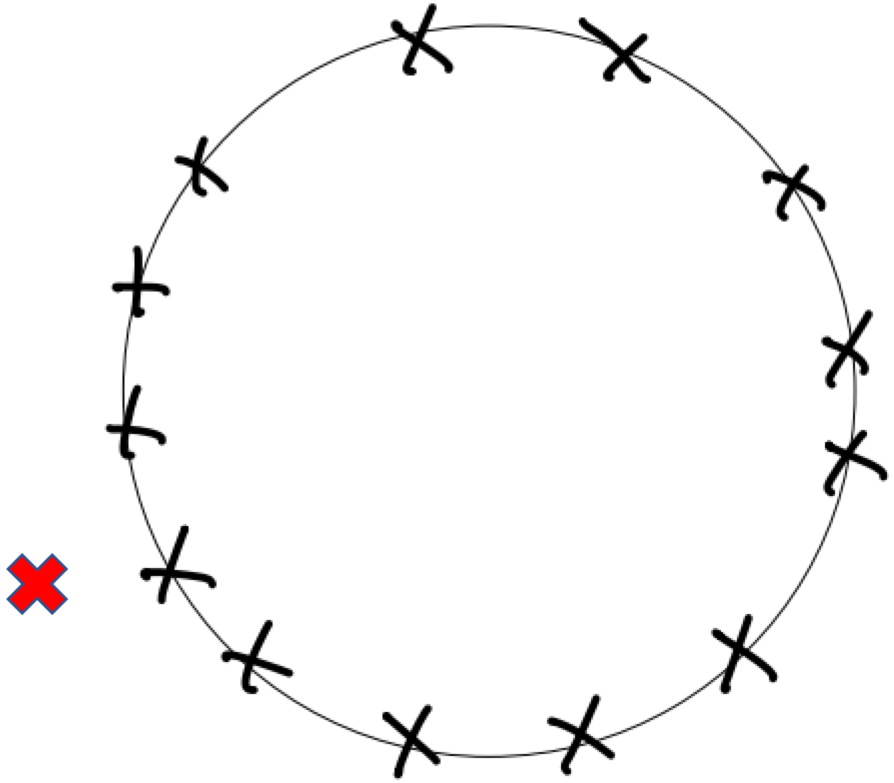

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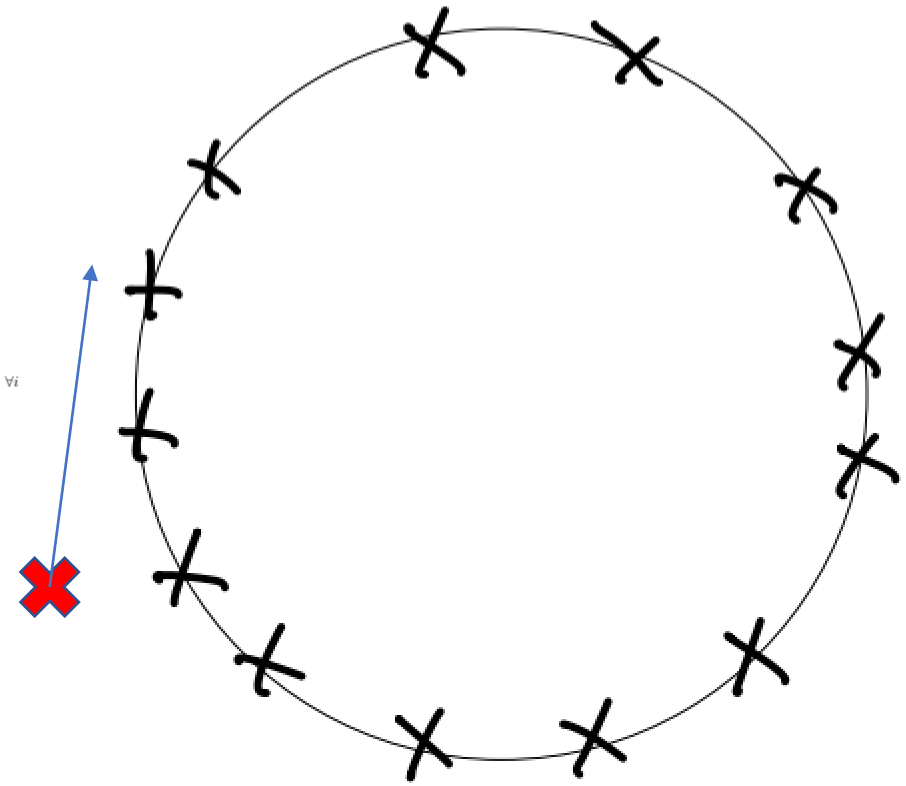
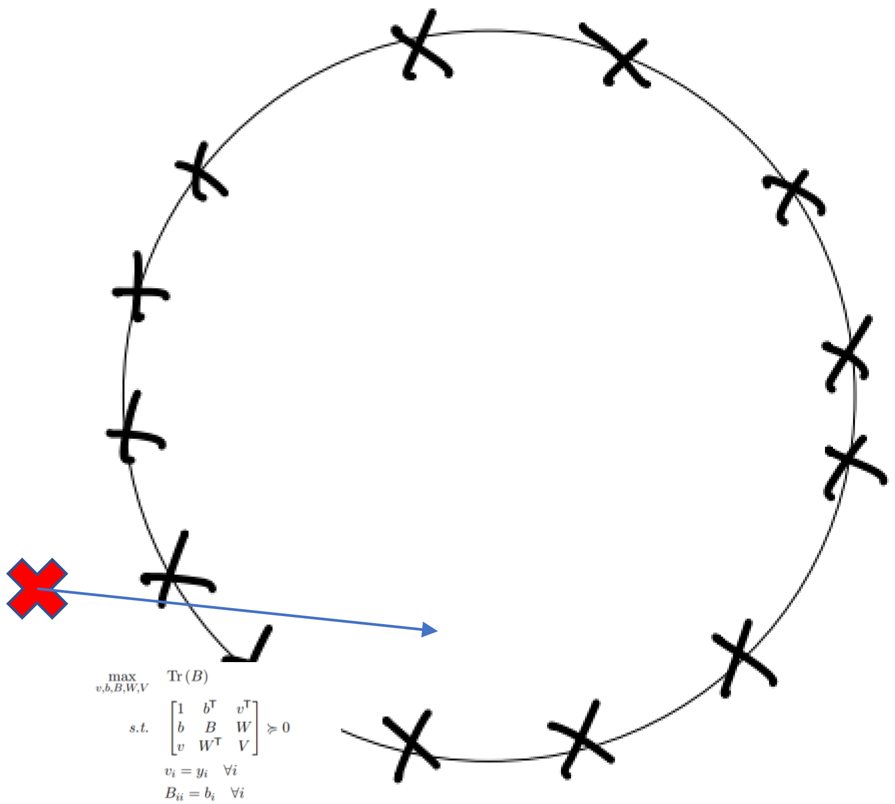


Run exponential mechanism!

Running efficient sampler…



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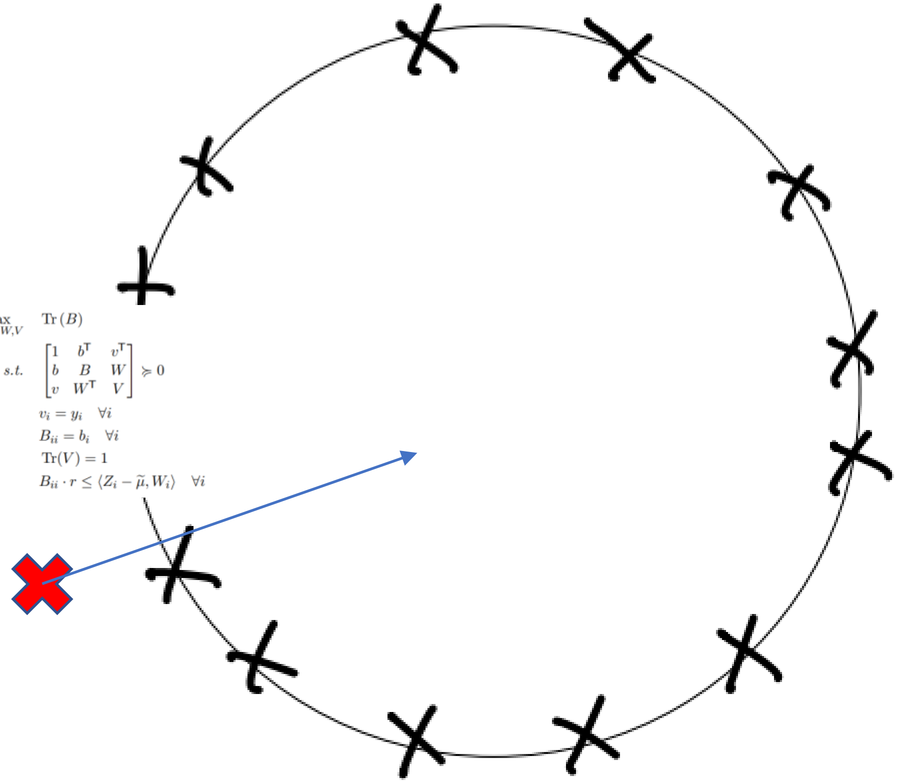

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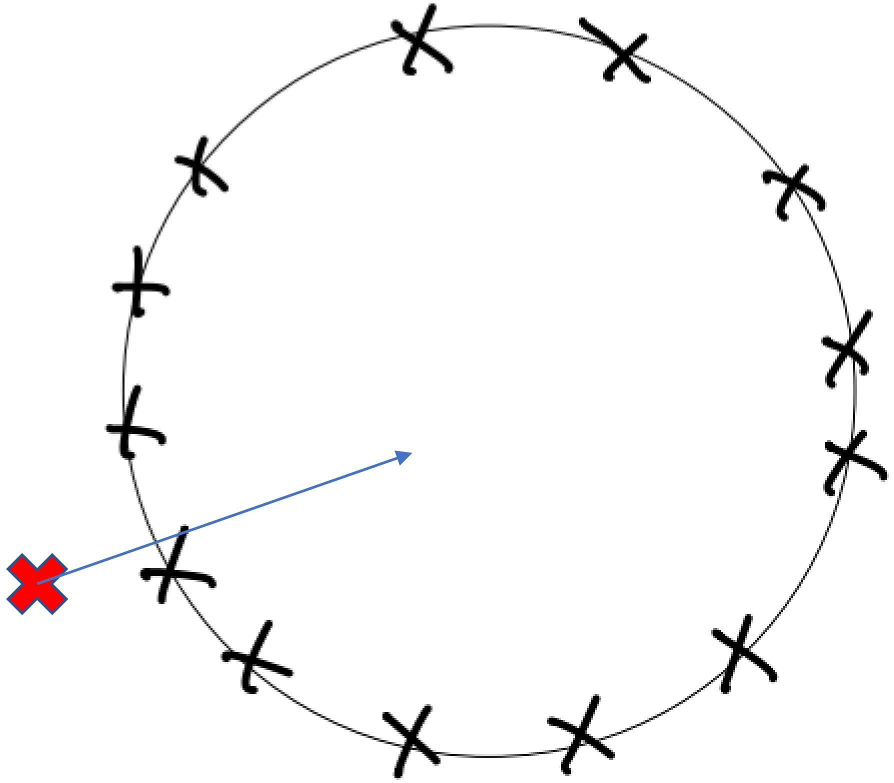
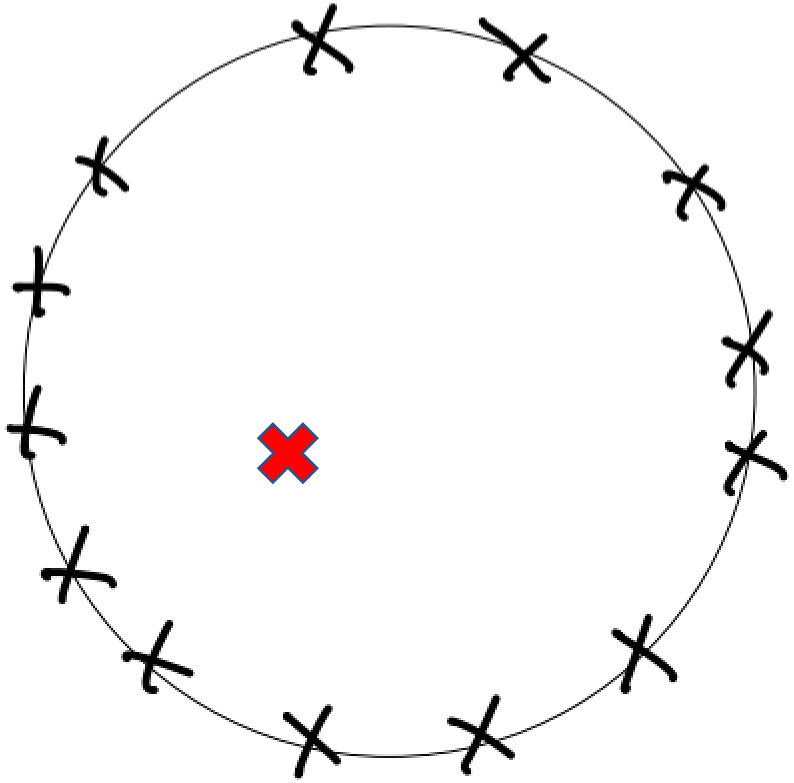


Etc.



A Meta-Theorem

Suppose we have a s … llowing properties:

1. High score funct
2. Bounded sensit
3. Lipschitz wrt do
4. Concave wrt do
5. Large volume of … main of candidates

Then exp. mech. wil … y point efficiently.

We give a specific m … score functions

Framework also works for "coarse estimation" (not mentioned today)

Theorem 4.5 (Meta-Theorem on SoS Exponential Mechanism). *Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a compact, convex set and $\mathcal{X}$ a universe of possible datasets, equipped with a "neighbors" relation. Suppose that for every dataset X there exists an Archimedean and η-robustly satisfiable system of polynomial inequalities $\mathcal{P}^X(x,y) = \{p_1^X(x,y) \geq 0, \ldots, p_N^X(x,y) \geq 0\}$ and a polynomial $p^X(x,y)$, all of degree at most D, in indeterminates $x_1, \ldots, x_n, y_1, \ldots, y_N$ such that for every neighboring dataset X' there is a linear function $y'(y)$ such that bounded sensitivity has an SoS proof:*

$$\forall j,\ \mathcal{P}^X(x,y) \vdash_{\deg(p_j^{X'})} p_j^{X'}(x, y'(y)) \text{ and } \mathcal{P}^X(x,y) \vdash_D p^X(x,y) - p^{X'}(x,y') \leq 1\,.$$

Suppose also that for every X, there are SoS proofs $\mathcal{P}^X(x,y) \vdash_D p(x,y) \leq 1/\eta$ and $\mathcal{P}^X(x,y) \vdash_D -p(x,y) \leq 1/\eta$. Furthermore, suppose that the polynomials $\mathcal{P}^X$ and p^X, and the polynomials used in the above SoS proofs, all have coefficients expressible in at most B bits.

Then for every $\varepsilon > 0$ and $D \in \mathbb{N}$ there exists an ε-differentially private algorithm which takes as input the polynomials $p^X, p_1^X, \ldots, p_m^X$ and $B, \eta > 0$, with the following guarantees:

Utility: *For every X, if there is an SoS proof of utility for X which is degree-$\deg(p)$ with respect to p, i.e.,*

$$\mathcal{P}^X(x,y) \cup \{p(x,y) \geq 0\} \vdash_{D, \deg_p = \deg(p)} \|x - x^*(X)\|^2 \leq \alpha^2$$

for some vector $x^*(X) \in \mathbb{R}^n$ and $\alpha > 0$, where the coefficients of all polynomials involved in the proof are expressible with B bits, and if

$$\frac{\mathrm{vol}(\mathcal{C})}{\mathrm{vol}(\{x \in \mathcal{C} : \exists y \text{ s.t. } \mathcal{P}^X(x,y) \text{ and } p^X(x,y) \geq t\})} \leq r\,,$$

then the algorithm outputs x such that $\|x - x^*(X)\| \leq \alpha + 2^{-B}$ with probability at least $1 - r\exp(-\Omega(\varepsilon t))$.

Running time: *The algorithm runs in time*

$$\mathrm{poly}\left(n^D, N^D, m^D, \frac{1}{\varepsilon}, \frac{1}{\eta}, \mathrm{diam}(\mathcal{C}), B\right),$$

making at most this many calls to membership and projection oracles for $\mathcal{C}$.

Overall Theorem

Given an η-corrupted set of samples $X_1, \ldots, X_n \sim p$ where p has covariance $\|\Sigma\|_2 \leq 1$ and $\|E[p]\|_2 \leq R$, there exists a computationally efficient $(\varepsilon, 0)$-DP algorithm which outputs $\hat{\mu}$ such that $\|\hat{\mu} - E[p]\|_2 \leq \alpha + O\left(\sqrt{\eta}\right)$ with probability $1 - \beta$. It requires

$$n = \tilde{O}\left(\frac{d+\log(1/\beta)}{\alpha^2 \varepsilon} + \frac{d \log R + \min(d, \log R) \log(1/\beta)}{\varepsilon}\right) \text{ samples.}$$

No algorithm can succeed with fewer than

$$n = \Omega\left(\frac{d+\log(1/\beta)}{\alpha^2 \varepsilon} + \frac{d \log R + \log(1/\beta)}{\varepsilon}\right) \text{ samples.}$$

[Hopkins, K., Majid], STOC 2022

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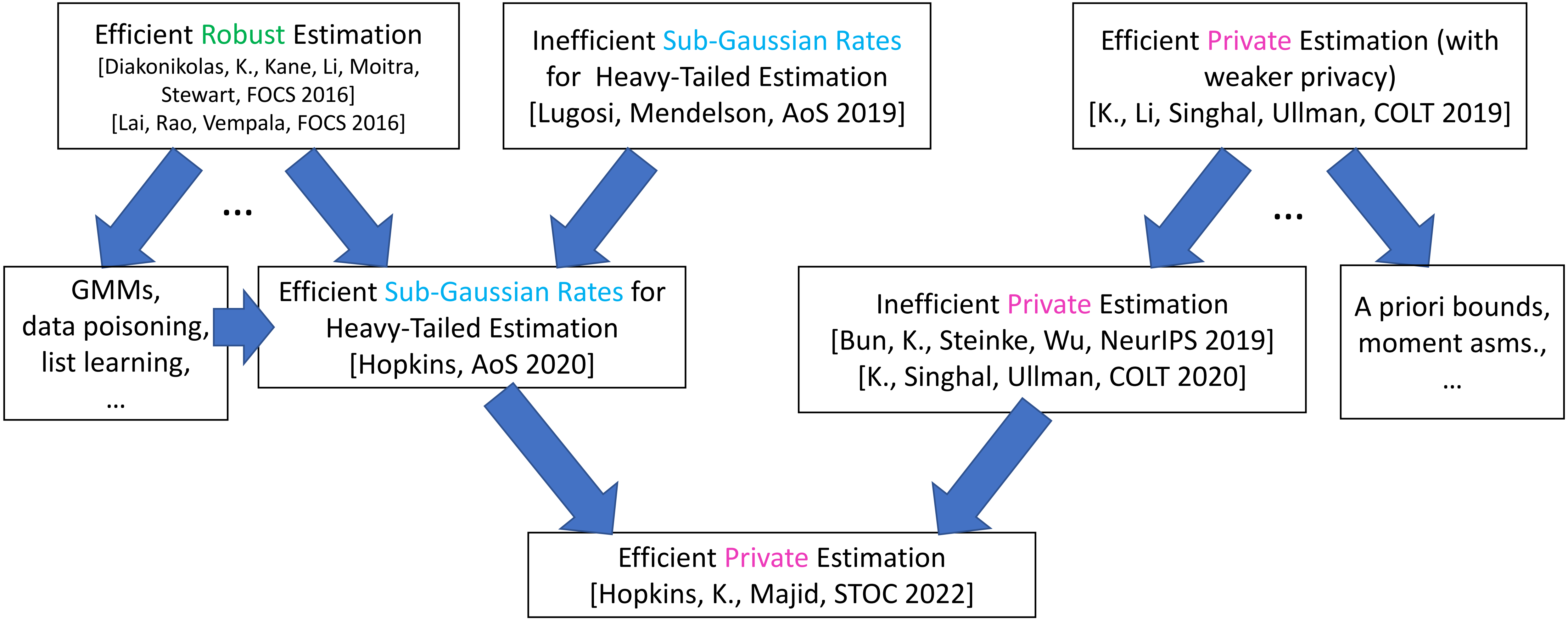
More importantly: new results for Gaussians (including covariance)

[Hopkins, K., Majid, Narayanan], 2022

The plan from here…

1. Three Deficient Private Algorithms
2. Fixing the Exponential Mechanism
3. The Algorithm
4. The Bigger Picture



Robustness, sub-Gaussian rates, and privacy:

All connected by the same technical ideas!

Are Robust and Private Estimation Equivalent?

- Privacy implies robustness…
  - As long as the private algorithm has a very high success probability
  - [Georgiev, Hopkins, NeurIPS 2022]
- Robustness implies privacy…
  - Assuming the “good” solutions have a large enough volume
  - [Hopkins, K., Majid, Narayanan, 2022]
- Close… but still some (significant) gaps

Conclusion

- First efficient pure DP algorithm with $\tilde{O}(d)$ sample complexity for mean estimation
- Also gets robustness and sub-Gaussian tails for free!

Open directions:

- More connections between robust and private estimation?
- Where else can the powerful SoS framework be used for DP estimation?