AI4OPT Tutorial Series: Practice problems

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1. Let $\{P_n : n \geq 1\}$ and $\{Q_n : n \geq 1\}$ be probability measures on $(\mathcal{X}_n, \mathcal{F}_n)$. Prove that the following are equivalent:

   - Q_n is contiguous to P_n.
   - For any sequence of real valued random variables T_n, if $T_n \xrightarrow{P_n} 0$, then $T_n \xrightarrow{Q_n} 0$.

2. Let $\{P_n : n \geq 1\}$ and $\{Q_n : n \geq 1\}$ be probability measures on $(\mathcal{X}_n, \mathcal{F}_n)$. If $\mathbb{E}_{P_n}\left[\left(\frac{\mathrm{d}Q_n}{\mathrm{d}P_n}\right)^2\right]$ exists and remains bounded as $n \to \infty$, then $\{Q_n : n \geq 1\}$ is contiguous to $\{P_n : n \geq 1\}$.

3. Let σ, τ be iid samples from $\mathrm{Unif}(\{\pm 1\}^n)$. Fix $\lambda < 1$. Prove that there exists $M > 0$ (independent of n) such that

$$\limsup_{n \to \infty} \mathbb{E}\Big[\exp\Big(\frac{\lambda^2}{2n}\langle \sigma, \tau \rangle^2\Big)\Big] \leq M.$$

4. Fix $\lambda, \mu > 0$. Let $\mathbf{W} \in \mathbb{R}^{n \times n}$, $\mathbf{W} = \mathbf{W}^\top$, $\{W_{ij} : i < j\} \sim \mathcal{N}(0, 1/n)$ iid, and $\{W_{ii} : 1 \leq i \leq n\} \sim \mathcal{N}(0, 2/n)$ iid. Let σ be a uniform random sample from $\{\pm 1\}^n$ and $u \sim \mathcal{N}(0, I_p/p)$. Finally, let $\mathbf{A} = \frac{\lambda}{n}\sigma\sigma^\top + \mathbf{W}$, and $\mathbf{B} = [b_1, \cdots, b_n]$, $b_i = \sqrt{\frac{\mu}{n}}\sigma_i u + z_i$, where $z_i \sim \mathcal{N}(0, I_p/p)$ are iid. Assume that $p/n \to 1/\gamma \in (0, \infty)$. This is a gaussian version of the contextual stochastic block model. Prove that consistent detection is impossible if $\lambda^2 + \frac{\mu^2}{\gamma} < 1$.