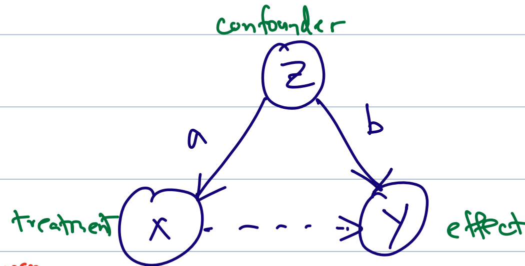
Linear Regression and Adjustment

Note: This notion of "adjustment" or "control" is from linear regression. Suppose (X, Y, Z) are jointly Gaussian, and zero-mean.



Ground-truth model

this is zero, i.e., no true effect of X on Y

$$Z = N_1 \sim N(0,1)$$
$$X = aZ + N_2$$
$$Y = bZ + N_3$$

$$\min_{\gamma, \beta} E\left[\left(Y - \gamma X - \beta Z\right)^2\right]$$

Use least squares to regress Y on (X, Z):

Solve for γ, β using LMS:

N_1, N_2, N_3 indep. $\sim N(0,1)$.

$$E\left[-2X\left(Y - \gamma X - \beta Z\right)\right] = 0$$
$$E\left[-2Z\left(Y - \gamma X - \beta Z\right)\right] = 0$$

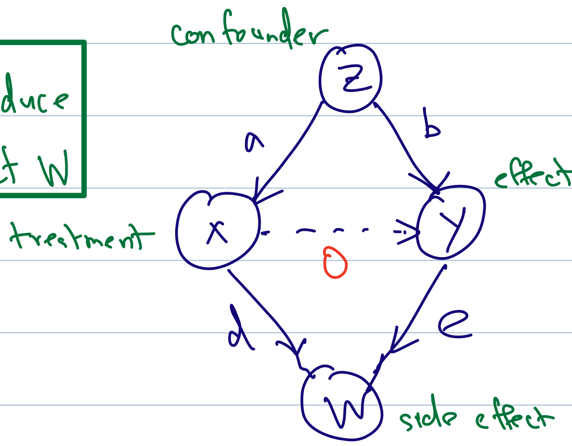
$$E[XY] = \gamma E[X^2] + \beta E[XZ].$$
$$E[ZY] = \gamma E[XZ] + \beta E[Z^2].$$

$$ab = \gamma(a^2+1) + \beta(a)$$
$$b = \gamma(a) + \beta(1).$$

$$\boxed{\beta = b. \qquad \gamma = 0}$$

Takeaway: Linear regression indeed sets $\gamma = 0$ (meaning the regression correctly shows that treatment X does not influence the effect Y when adjusting with Z)

Now introduce side effect W



Ground-truth model)

$$Z = N_1$$
$$X = aZ + N_2$$
$$Y = bZ + N_3$$
$$W = dX + eY + N_4.$$

Now suppose I regress Y on (X, W, Z), i.e

$$\min_{\gamma,\beta,\phi} E\left[\left(Y - \gamma X - \beta Z - \phi W\right)^2\right]$$

$$E[XY] = \gamma E[X^2] + \beta E[XZ] + \phi E[XW]$$
$$E[ZY] = \gamma E[XZ] + \beta E[Z^2] + \phi E[WZ]$$
$$E[WY] = \gamma E[XW] + \beta E[ZW] + \phi E[W^2].$$

i.e. $$ab = \gamma(a^2+1) + \beta(a) + \phi\big(d(a^2+1) + e(ab)\big).$$
$$b = \gamma(a) + \beta(1) + \phi\big(d(a) + e(b)\big)$$
$$bd(a) + e(b^2+1) = \gamma(ab) + \beta(ad+be) + \phi\big(d^2(a^2+1) + e^2(b^2+1) + 1 + de(ab)\big)$$

Check: Ideally, we want: $\gamma = 0$, $\beta = b$, $\phi = 0$ because here W is a "side effect"; the only causal predictor of Y is Z.

$$ab = \cancel{\gamma(a^2+1)} + \overset{=b}{\beta}(a) + \cancel{\phi\big(d(a^2+1) + e(ab)\big)}.$$
$$b = \cancel{\gamma(a)} + \overset{=b}{\beta}(1) + \cancel{\phi\big(d(a) + e(b)\big)}$$
$$bd(a) + e(b^2+1) = \cancel{\gamma(ab)} + \overset{=b}{\beta}(ad+be) + \cancel{\phi\big(d^2(a^2+1) + e^2(b^2+1) + 1 + de(ab)\big)}$$

this is not satisfied! (there is an extra 'e' in the LHS).

$\Longrightarrow$ $\boxed{\text{Adding extra covariates introduces spurious dependencies}}$

Summary: Adding extra variables can lead to seeming dependencies between "cause" and "effect", despite there being no such relationship in the ground truth.